1.

The graph of $y = f(x)$ is transformed to the graph of $y = 3 - f(x)$.

Describe fully, in the correct order, the two transformations that have been combined. [4]

$y = \underbrace{-f(x)}_{} + \underbrace{3}_{}$

1. Reflection in x axis

2. Translation along y axis with vector $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

2.

(a) Find the first three terms, in ascending powers of x, in the expansion of $(1 + ax)^6$. [1]

$1^6 + {}^6C_1 (1)^{6-1} (ax)^1 + {}^6C_2 (1)^{6-2} (ax)^2$

$1 + 6ax + 15a^2x^2$

Formula: $(a+b)^n = a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \dots$

(b) Given that the coefficient of x^2 in the expansion of $(1 - 3x)(1 + ax)^6$ is -3, find the possible values of the constant a. [4]

$(1-3x)(1+ax)^6 \longrightarrow x^2$

$(1-3x)(1+6ax+15a^2x^2)$

$1 \times 15a^2x^2 = 15a^2x^2$

$-3x \times 6ax = -18ax^2$

$15a^2 - 18a = -3$

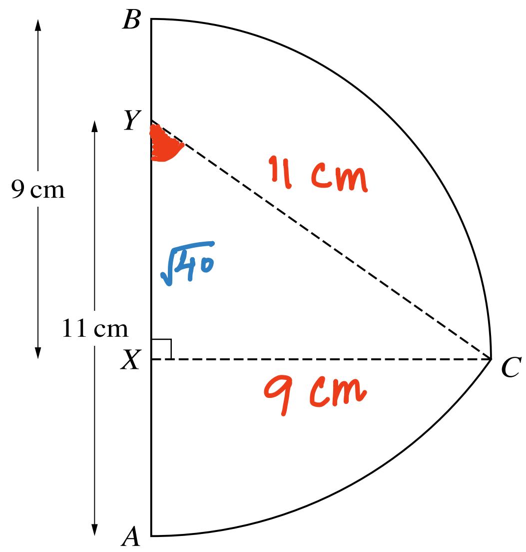
$15a^2 - 18a + 3 = 0$

$5a^2 - 6a + 1 = 0$

$(a-1)(5a-1) = 0$

$\boxed{a = 1, \quad a = \frac{1}{5}}$

3.



In the diagram, X and Y are points on the line AB such that $BX = 9$ cm and $AY = 11$ cm. Arc BC is part of a circle with centre X and radius 9 cm, where CX is perpendicular to AB. Arc AC is part of a circle with centre Y and radius 11 cm.

(a) Show that angle $XYC = 0.9582$ radians, correct to 4 significant figures. [1]

$\sin(xyc) = \frac{9}{11}$

$x\hat{y}c = \sin^{-1}\left(\frac{9}{11}\right)$

$x\hat{y}c = 0.9582$ radians

(b) Find the perimeter of ABC. [6]

$XY = \sqrt{11^2 - 9^2} = \sqrt{40}$

$AB = BX + XA$

$AB = 9 + AY - YX$

$AB = 9 + 11 - \sqrt{40}$

$AB = 20 - \sqrt{40}$

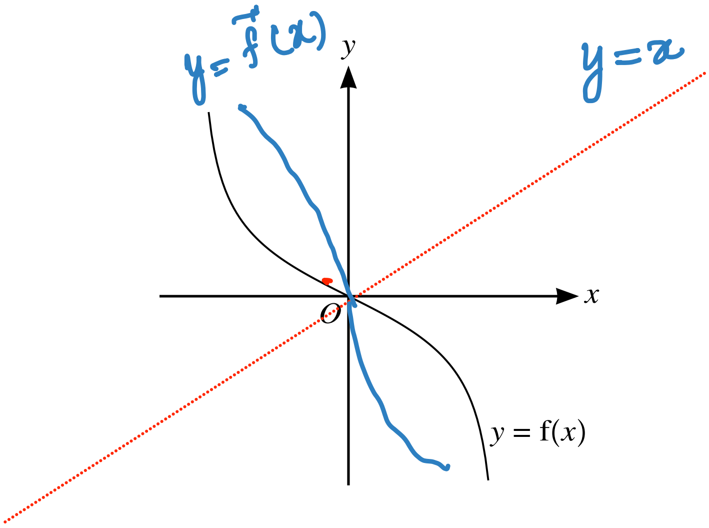
Perimeter of ABC = Arc length of BC + arc length of CA + AB

$\boxed{\text{Arc length} = r\theta}$

$= \frac{\pi}{2} \times 9 + 0.9582 \times 11 + 20 - \sqrt{40}$

$= 38.4$

4.



The diagram shows the graph of $y = f(x)$.

(a) On this diagram sketch the graph of $y = f^{-1}(x)$. [1]

It is now given that $f(x) = -\dfrac{x}{\sqrt{4-x^2}}$ where $-2 < x < 2$.

(b) Find an expression for $f^{-1}(x)$. [4]

$$y = \frac{-x}{\sqrt{4-x^2}}$$

$$y^2 = \frac{x^2}{4-x^2}$$

$$4y^2 - x^2y^2 = x^2$$

$$x^2 + x^2y^2 = 4y^2$$

$$x^2(1+y^2) = 4y^2$$

$$x^2 = \frac{4y^2}{1+y^2}$$

$$x = \sqrt{\frac{4y^2}{1+y^2}}$$

$x = \pm \dfrac{2y}{\sqrt{1+y^2}}$

$f^{-1}(x) = \dfrac{2x}{\sqrt{1+x^2}}$

$f^{-1}(x) = -\dfrac{2x}{\sqrt{1-x^2}}$

To check the sign:
Denominator is always positive.

So, sign of inverse function depends on the numerator x-value.

or check $f(1) = -1/\sqrt{3}$

$f^{-1}(-1/\sqrt{3}) = 1$

The function g is defined by $g(x) = 2x$ for $-a < x < a$, where a is a constant.

(c) State the maximum possible value of a for which fg can be formed. [1]

Range of $g(x)$: $-2a < g(x) < 2a$

Domain of $f(x)$: $-2 < x < 2$ (given)

$2a = 2$

$\boxed{a = 1}$

f

(d) Assuming that fg can be formed, find and simplify an expression for $fg(x)$. [2]

$f(x) = \dfrac{-x}{\sqrt{4-x^2}}$ $\quad g(x) = 2x$

$fg(x) = \dfrac{-(2x)}{\sqrt{4-(2x)^2}}$

$fg(x) = \dfrac{-2x}{\sqrt{4-4x^2}} = \dfrac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}$

$\boxed{fg(x) = \dfrac{-x}{\sqrt{1-x^2}}}$

5.

(a) Show that the equation $\frac{\tan x + \cos x}{\tan x - \cos x} = k$, where k is a constant, can be expressed as

$$(k+1)\sin^2 x + (k-1)\sin x - (k+1) = 0. \qquad [4]$$

$$\frac{\frac{\sin x}{\cos x} + \cos x}{\frac{\sin x}{\cos x} - \cos x} = k$$

$$\frac{\frac{\sin x + \cos^2 x}{\cancel{\cos x}}}{\frac{\sin x - \cos^2 x}{\cancel{\cos x}}} = k$$

$$\frac{\sin x + \cos^2 x}{\sin x - \cos^2 x} = k$$

$$\sin x + \cos^2 x = k\sin x - k\cos^2 x$$

$$k\sin x - \sin x - \cos^2 x - k\cos^2 x = 0$$

$$\sin x(k-1) + \cos^2 x(-1-k) = 0$$

$$\sin x(k-1) + (1-\sin^2 x)(-1-k) = 0$$

$$\sin x(k-1) - 1 - k + \sin^2 x + k\sin^2 x = 0$$

$$\sin^2 x(k+1) + \sin x(k-1) - (k+1) = 0$$

(b) Hence solve the equation $\dfrac{\tan x + \cos x}{\tan x - \cos x} = 4$ for $0° \leqslant x \leqslant 360°$. [4]

$K = 4$

so, $\sin^2 x(K+1) + \sin x(K-1) - (K+1) = 0$

$5\sin^2 x + 3\sin x - 5 = 0$

$y = \sin x$

$5y^2 + 3y - 5 = 0$

$$y = \frac{-3 \pm \sqrt{3^2 - 4(5)(-5)}}{2(5)}$$

$$y = \frac{-3 \pm \sqrt{109}}{10}$$

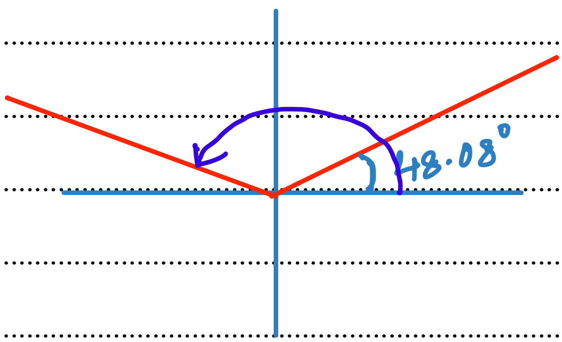
$y = \dfrac{-3 + \sqrt{109}}{10}$ $\qquad$ $y = \dfrac{-3 - \sqrt{109}}{10}$

$\sin x = 0.74403$ $\qquad$ $\sin x = -1.34403$

$x = \sin^{-1}(0.74403)$ $\qquad$ (NOT POSSIBLE)

$x = 48.08°$

$x = 180 - 48.08 = 131.92$



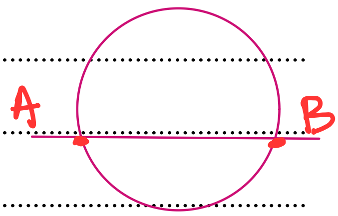
$x = 48.08°, \; 131.92°$

6.

The line $y = 2x + 5$ intersects the circle with equation $x^2 + y^2 = 20$ at A and B.

(a) Find the coordinates of A and B in surd form and hence find the exact length of the chord AB. [7]

$y = 2x + 5$ $\qquad$ $x^2 + y^2 = 20$



$$x^2 + (2x+5)^2 = 20$$
$$x^2 + 4x^2 + 20x + 25 = 20$$
$$5x^2 + 20x + 25 - 20 = 0$$
$$5x^2 + 20x + 5 = 0$$
$$x^2 + 4x + 1 = 0$$

$$x = \frac{-4 \pm \sqrt{4^2 - 4\times1\times1}}{2}$$

$$x = \frac{-4 \pm 2\sqrt{3}}{2}$$

$$x = \frac{-4 + 2\sqrt{3}}{2} \qquad x = \frac{-4 - 2\sqrt{3}}{2}$$

$$x = \frac{\cancel{2}[-2+\sqrt{3}]}{\cancel{2}} \qquad x = \frac{\cancel{2}[-2-\sqrt{3}]}{\cancel{2}}$$

$$x = -2+\sqrt{3} \qquad x = -2-\sqrt{3}$$

$y = 2x + 5$ $\qquad$ $y = 2x + 5$

$y = 2(-2+\sqrt{3}) + 5$ $\qquad$ $y = 2(-2-\sqrt{3}) + 5$

$y = -4 + 2\sqrt{3} + 5$ $\qquad$ $y = -4 - 2\sqrt{3} + 5$

$y = 1 + 2\sqrt{3}$ $\qquad$ $y = 1 - 2\sqrt{3}$

$(-2+\sqrt{3},\ 1+2\sqrt{3})$ $\qquad$ $(-2-\sqrt{3},\ 1-2\sqrt{3})$

length of AB $= \sqrt{(-2-\sqrt{3}+2-\sqrt{3})^2 + (1-2\sqrt{3}-1-2\sqrt{3})^2}$

$= \sqrt{(-2\sqrt{3})^2 + (-4\sqrt{3})^2}$

$= \sqrt{12+48} = \sqrt{60}$

A straight line through the point (10, 0) with gradient m is a tangent to the circle.

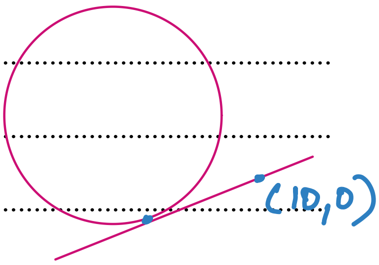
(b) Find the two possible values of m. [5]

Equation of a tangent

$$y - y_1 = m(x - x_1)$$

$$y - 0 = m(x - 10)$$

$$y = mx - 10m$$



Equation of Circle

$$x^2 + y^2 = 20$$

$$x^2 + (mx - 10m)^2 = 20$$

$$x^2 + m^2x^2 - 20m^2x + 100m^2 = 20$$

$$x^2(1 + m^2) - 20m^2x - 20 + 100m^2 = 0$$

For Tangency

$$b^2 - 4ac = 0$$

$$(20m^2)^2 - 4(1+m^2)(-20+100m^2) = 0$$

$$400m^4 - 4[-20 + 100m^2 - 20m^2 + 100m^4] = 0$$

$$\cancel{400m^4} + 80 - 400m^2 + 80m^2 - \cancel{400m^4} = 0$$

$$-320m^2 + 80 = 0$$

$$-320m^2 = -80$$

$$m^2 = \frac{80}{320}$$

$$m^2 = \frac{1}{4}$$

$$\boxed{m = \pm\frac{1}{2}}$$

7.

The equation of a curve is $y = 2x^2 + kx + k - 1$, where k is a constant.

(a) Given that the line $y = 2x + 3$ is a tangent to the curve, find the value of k. [3]

The line touches the curve at one point.

$y = 2x+3 \qquad y = 2x^2+kx+k-1$

$2x^2+kx+k-1 = 2x+3$

$2x^2+kx-2x+k-1-3=0$

$2x^2+(k-2)x+(k-4)=0$

For tangency : $b^2-4ac=0$

$(k-2)^2-4\times2\times(k-4)=0$

$k^2-4k+4-8(k-4)=0$

$k^2-4k+4-8k+32=0$

$k^2-12k+36=0$

$(k-6)(k-6)=0 \qquad \boxed{k=6}$

It is now given that $k = 2$.

(b) Express the equation of the curve in the form $y = 2(x + a)^2 + b$, where a and b are constants, and hence state the coordinates of the vertex of the curve. [3]

$y = 2x^2+kx+k-1$

$y = 2x^2+2x+2-1$

$y = 2x^2+2x+1$

$y = 2(x^2+x)+1$

$y = 2\left[\left(x+\tfrac{1}{2}\right)^2-\left(\tfrac{1}{2}\right)^2\right]+1$

$y = 2\left[\left(x+\tfrac{1}{2}\right)^2-\tfrac{1}{4}\right]+1$

$y = 2\left(x+\tfrac{1}{2}\right)^2-\tfrac{1}{2}+1$

$y = 2\left(x+\tfrac{1}{2}\right)^2+\tfrac{1}{2}$

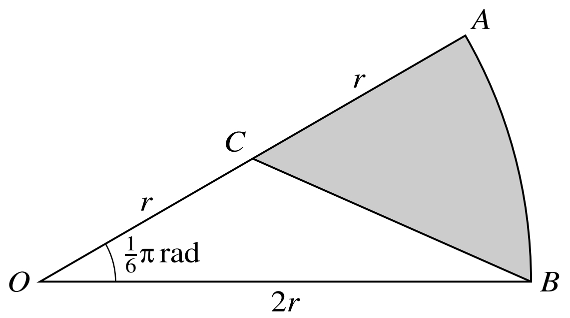
Vertex

stationary

Turning Points

$\left(-\tfrac{1}{2}, \tfrac{1}{2}\right)$

8.



In the diagram, OAB is a sector of a circle with centre O and radius $2r$, and angle $AOB = \frac{1}{6}\pi$ radians. The point C is the midpoint of OA.

(a) Show that the exact length of BC is $r\sqrt{5 - 2\sqrt{3}}$. [2]

Applying Cosine Rule in Triangle OBC

$$BC^2 = r^2 + (2r)^2 - 2 \times r \times 2r \times \cos(\tfrac{1}{6}\pi)$$

$$BC^2 = r^2 + 4r^2 - 4r^2 \times \sqrt{3}/2$$

$$BC^2 = 5r^2 - 2\sqrt{3}\,r^2$$

$$BC^2 = r^2[5 - 2\sqrt{3}]$$

$$BC = \sqrt{r^2[5 - 2\sqrt{3}]}$$

$$BC = r\sqrt{5 - 2\sqrt{3}}$$

Arc length $= r\theta$

(b) Find the exact perimeter of the shaded region. [2]

Perimeter $= CA + \overset{\frown}{AB} + BC$

$= r + 2r \cdot \frac{1}{6}\pi + r\sqrt{5-2\sqrt{3}}$

Perimeter $= r + \frac{r}{3}\pi + r\sqrt{5-2\sqrt{3}}$

(c) Find the exact area of the shaded region. [3]

Area of Sector $OAB = \frac{1}{2}r^2\theta$

$= \frac{1}{2}(2r)^2 \; \frac{1}{6}\pi$

$= \frac{1}{2} \times 4r^2 \times \frac{1}{6}\pi$

$= \frac{r^2}{3}\pi$

Area of triangle $OCB = \frac{1}{2} \times r \times 2r \times \sin\left(\frac{1}{6}\pi\right)$

$= \frac{1}{2} \times 2r^2 \times \frac{1}{2}$

$= \frac{r^2}{2}$

$\boxed{\text{Area of shaded} = \frac{r^2}{3}\pi - \frac{r^2}{2}}$

9.

Functions f and g are such that

$$f(x) = 2 - 3\sin 2x \quad \text{for } 0 \leqslant x \leqslant \pi,$$
$$g(x) = -2f(x) \quad \text{for } 0 \leqslant x \leqslant \pi.$$

(a) State the ranges of f and g. [3]

$-1 \leq \sin 2x \leq 1$

$f(x) = 2 - 3(-1) = 5$

$f(x) = 2 - 3(1) = -1$

$g(x) = -2(5) = -10$

$g(x) = -2(-1) = 2$

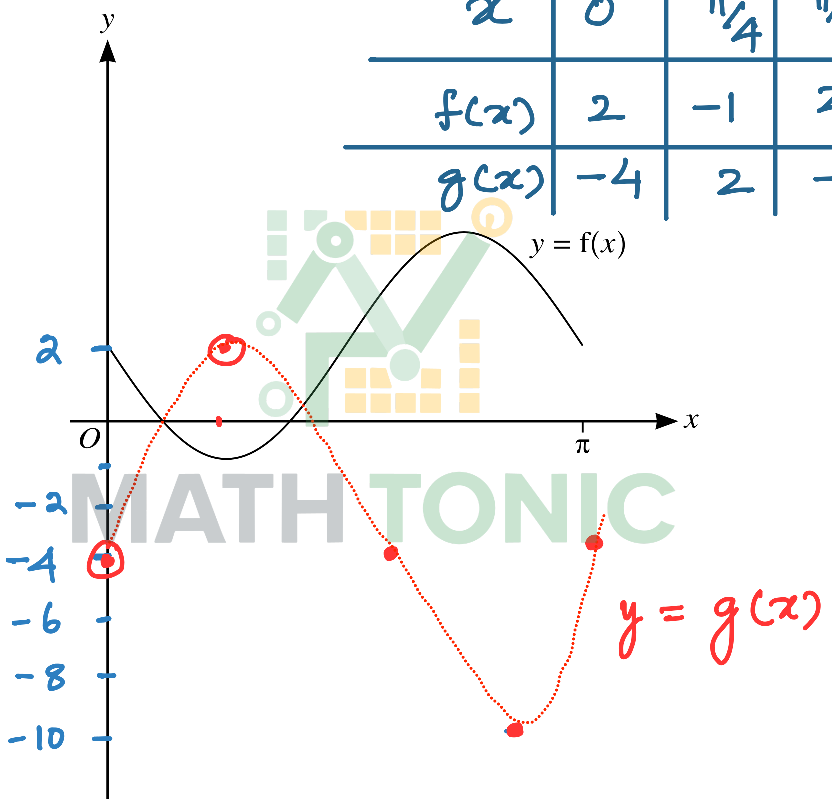
$-1 \leq f(x) \leq 5$

$-10 \leq g(x) \leq 2$

Ranges

The diagram below shows the graph of $y = f(x)$.

x	0	$\pi/4$	$\pi/2$	$3\pi/2$	π
$f(x)$	2	-1	2	5	2
$g(x)$	-4	2	-4	-10	-4



(b) Sketch, on this diagram, the graph of $y = g(x)$. [2]

The function h is such that

$$h(x) = g(x + \pi) \quad \text{for } -\pi \leqslant x \leqslant 0.$$

(c) Describe fully a sequence of transformations that maps the curve $y = f(x)$ on to $y = h(x)$. [3]

$f(x) = 2 - 3\sin 2x$ * Reflection in x-axis

$g(x) = -2(f(x))$ * Stretched along y-direction with factor 2

$h(x) = g(x + \pi)$ * Translation by $\begin{pmatrix} 0 \\ -\pi \end{pmatrix}$

10.

The equation of a circle with centre C is $x^2 + y^2 - 8x + 4y - 5 = 0$.

(a) Find the radius of the circle and the coordinates of C. [3]

$$x^2 - 8x + y^2 + 4y - 5 = 0$$
$$(x-4)^2 - 4^2 + (y+2)^2 - 2^2 - 5 = 0$$
$$(x-4)^2 - 16 + (y+2)^2 - 4 - 5 = 0$$
$$(x-4)^2 + (y+2)^2 - 25 = 0$$
$$(x-4)^2 + (y+2)^2 = 25$$
$$(x-4)^2 + (y+2)^2 = 5^2$$

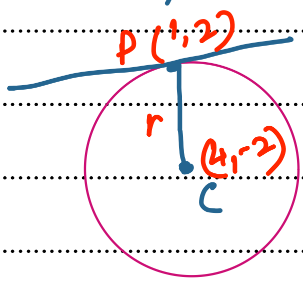
Radius is 5 and Centre $(4,-2)$

The point $P\,(1,\ 2)$ lies on the circle.

(b) Show that the equation of the tangent to the circle at P is $4y = 3x + 5$. [3]

Centre $(4,-2)$

Gradient of radius $= \dfrac{y_2-y_1}{x_2-x_1} = \dfrac{2-(-2)}{1-4} = -\dfrac{4}{3}$



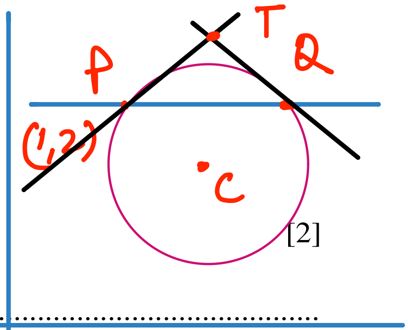
Gradient of tangent at P, $= \dfrac{3}{4}$

(because it is perpendicular)

Equation of tangent

$$y - y_1 = m(x - x_1)$$
$$y - 2 = \frac{3}{4}(x-1)$$
$$4y - 8 = 3x - 3$$
$$4y = 3x + 5$$

The point Q also lies on the circle and PQ is parallel to the x-axis.



(c) Write down the coordinates of Q. [2]

y coordinate of Q is 2

$Q(x, 2)$

Equation of Circle: $(x-4)^2+(y+2)^2=5^2$

$(x-4)^2+(2+2)^2=5^2$

$(x-4)^2=9$

$x-4=3 \Rightarrow x=7$

$Q(7,2)$

The tangents to the circle at P and Q meet at T.

(d) Find the coordinates of T. [3]

Gradient of CQ is $\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{2-(-2)}{7-4}=\dfrac{4}{3}$

Gradient of Tangent at Q is $-\dfrac{3}{4}$

Equation of tangent at Q is

$y-2=-\dfrac{3}{4}(x-7)$

$4y-8=-3x+21$

$4y=-3x+29$

$P: \; 4y=3x+5$

$Q: \; 4y=-3x+29$

Simultaneous equation

$3x+5=-3x+29$

$6x=24$

$x=\dfrac{24}{6}$

$x=4$

$4y=3x+5$

$4y=3\times4+5$

$4y=17$

$y=17/4$

$T(4, 17/4)$