1.

The coefficient of x^4 in the expansion of $(x+a)^6$ is p and the coefficient of x^2 in the expansion of $(ax+3)^4$ is q. It is given that $p+q=276$.

Find the possible values of the constant a. [4]

$$(x+a)^6 = x^6 + {}^6C_1 x^5 a + {}^6C_2 x^4 a^2 + \ldots$$
$$x^6 + 6x^5 a + 15x^4a^2 + \ldots$$

Coefficient of $x^4 = 15a^2$

$$p = 15a^2$$

$$(ax+3)^4 = (ax)^4 + {}^4C_1 (ax)^3(3) + {}^4C_2 (ax)^2 (3)^2 + \ldots$$
$$= a^4x^4 + 12a^3x^3 + 54a^2x^2$$

Coefficient of $x^2 = 54a^2$

$$q = 54a^2$$

$$p+q = 276$$

$$15a^2 + 54a^2 = 276$$
$$69a^2 = 276$$
$$a^2 = \frac{276}{69}$$
$$a^2 = 4$$

$$a = \sqrt{4}$$
$$a = \pm 2$$

MATH TONIC

2.

Solve the equation $8x^6 + 215x^3 - 27 = 0$. [3]

$$8x^6 + 215x^3 - 27 = 0$$

Let $x^3 = y$

$x^6 = y^2$

$$8y^2 + 215y - 27 = 0$$

$$y = \frac{-215 \pm \sqrt{(215)^2 - 4(8)(-27)}}{2(8)}$$

$y = \frac{1}{8}$ | $y = -27$

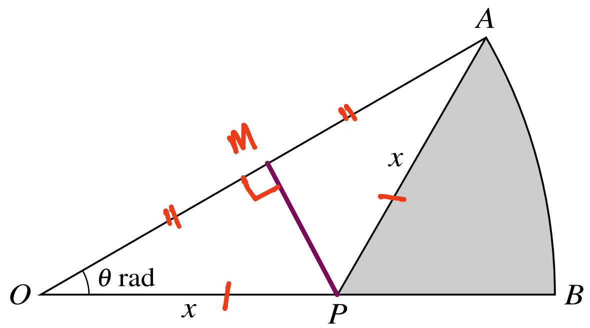
$x^3 = \frac{1}{8}$ | $x^3 = -27$

$x = \sqrt[3]{\frac{1}{8}}$ | $x = \sqrt[3]{-27}$

$x = \frac{1}{2}$ | $x = -3$

MATH TONIC

3.



The diagram shows a sector OAB of a circle with centre O. Angle $AOB = \theta$ radians and $OP = AP = x$.

(a) Show that the arc length AB is $2x\theta\cos\theta$. [2]

$\cos\theta = \frac{OM}{OP} = \frac{OM}{x}$

$OM = x\cos\theta$

As $OM = MA$

$OA = x\cos\theta + x\cos\theta$

$OA = 2x\cos\theta$

radius

Arc length $AB = r\theta$

$= 2x\cos\theta \times \theta$

AB Arc length $= 2x\theta\cos\theta$

(b) Find the area of the shaded region APB in terms of x and θ. [4]

Area of shaded = Area of sector OAB − Area of triangle OAP

$= \frac{1}{2}r^2\theta - \frac{1}{2}ab\sin C$

$= \frac{1}{2}(2x\theta\cos\theta)^2\theta - \frac{1}{2}x(2x\cos\theta)\sin\theta$

$= \frac{1}{2} \times 4x^2\cos^2\theta \times \theta - x^2\cos\theta \cdot \sin\theta$

$= 2x^2\theta\cos^2\theta - x^2\cos\theta \cdot \sin\theta$

$= x^2\cos\theta(2\theta\cos\theta - \sin\theta)$

MATH TONIC

4.

(a) **(i)** By first expanding $(\cos\theta + \sin\theta)^2$, find the three solutions of the equation

$$(\cos\theta + \sin\theta)^2 = 1$$

for $0 \leqslant \theta \leqslant \pi$. [3]

$$(\cos\theta + \sin\theta)^2 = 1$$

$$(\cos\theta + \sin\theta)(\cos\theta + \sin\theta) = 1$$

$$\cos^2\theta + \sin\theta\cdot\cos\theta + \sin\theta\cdot\cos\theta + \sin^2\theta = 1$$

$$\cos^2\theta + 2\sin\theta\cdot\cos\theta + \sin^2\theta = 1$$

$$\underline{\cos^2\theta + \sin^2\theta} + 2\sin\theta\cdot\cos\theta = 1$$

Rule: $\sin^2\theta + \cos^2\theta = 1$

$$1 + 2\sin\theta\cdot\cos\theta = 1$$

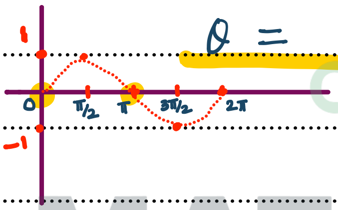
$$2\sin\theta\cdot\cos\theta = 0$$

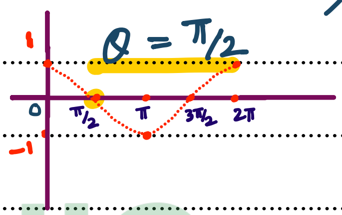
$$\sin\theta\cdot\cos\theta = 0$$

$\sin\theta = 0$ $\qquad\qquad$ $\cos\theta = 0$

$\theta = \sin^{-1}(0)$ $\qquad\qquad$ $\theta = \cos^{-1}(0)$

$\theta = 0, \pi$ $\qquad\qquad$ $\theta = \pi/2$





MATH TONIC

(ii) Hence verify that the only solutions of the equation $\cos\theta + \sin\theta = 1$ for $0 \leqslant \theta \leqslant \pi$ are 0 and $\frac{1}{2}\pi$. [2]

$$\cos\theta + \sin\theta = 1$$

For $x = 0$, $\quad \cos 0 + \sin 0 = 1$

For $x = \frac{\pi}{2}$, $\quad \cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 1$

For $x = \pi$, $\quad \cos\pi + \sin\pi = -1$

It is verified above that the only solutions of the equation $\cos\theta + \sin\theta = 1$ are 0 and $\frac{\pi}{2}$ only.

(b) Prove the identity $\dfrac{\sin\theta}{\cos\theta+\sin\theta}+\dfrac{1-\cos\theta}{\cos\theta-\sin\theta}\equiv\dfrac{\cos\theta+\sin\theta-1}{1-2\sin^2\theta}$. [3]

$$\frac{\sin\theta(\cos\theta-\sin\theta)+(1-\cos\theta)(\cos\theta+\sin\theta)}{(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)}$$

$$\frac{\cancel{\sin\theta\cdot\cos\theta}-\sin^2\theta+\cos\theta+\sin\theta-\cos^2\theta-\cancel{\sin\theta\cdot\cos\theta}}{\cos^2\theta-\cancel{\sin\theta\cdot\cos\theta}+\cancel{\sin\theta\cdot\cos\theta}-\sin^2\theta}$$

$$\frac{\cos\theta+\sin\theta-\sin^2\theta-\cos^2\theta}{\cos^2\theta-\sin^2\theta}$$

$$\frac{\cos\theta+\sin\theta-(\sin^2\theta+\cos^2\theta)}{(1-\sin^2\theta)-\sin^2\theta}$$

$$\frac{\cos\theta+\sin\theta-1}{1-2\sin^2\theta}$$

(c) Using the results of **(a)(ii)** and **(b)**, solve the equation

$$\frac{\sin\theta}{\cos\theta+\sin\theta}+\frac{1-\cos\theta}{\cos\theta-\sin\theta}=2(\cos\theta+\sin\theta-1)$$

for $0\leqslant\theta\leqslant\pi$. [3]

MATH TONIC

$$\frac{\cos\theta+\sin\theta-1}{1-2\sin^2\theta}=2(\cos\theta+\sin\theta-1)$$

$\cos\theta+\sin\theta-1=2(\cos\theta+\sin\theta-1)(1-2\sin^2\theta)$

$(\cos\theta+\sin\theta-1)-2(\cos\theta+\sin\theta-1)(1-2\sin^2\theta)=0$

$(\cos\theta+\sin\theta-1)[1-2(1-2\sin^2\theta)]=0$

$(\cos\theta+\sin\theta-1)(1-2+4\sin^2\theta)=0$

$(\cos\theta+\sin\theta-1)(4\sin^2\theta-1)=0$

$\cos\theta+\sin\theta-1=0$ $4\sin^2\theta-1=0$

from a(i) $\sin^2\theta=\frac{1}{4}$

$\theta=0,\ \pi/2$ $\sin\theta=\sqrt{1/4}$

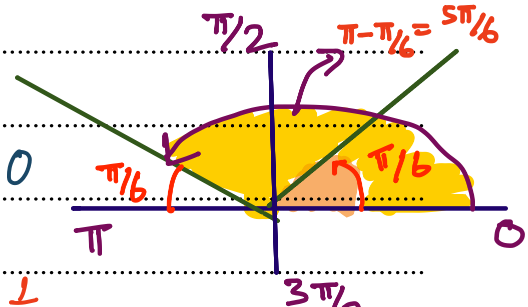
$\sin\theta=\frac{1}{2}$

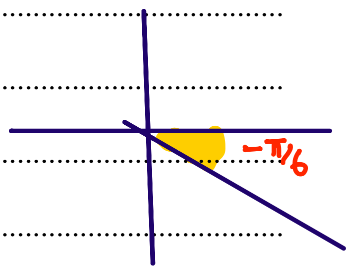
$\theta=\sin^{-1}(\frac{1}{2})$

$\theta=\frac{\pi}{6},\ \frac{5\pi}{6}$

$\sin\theta=-1/2$

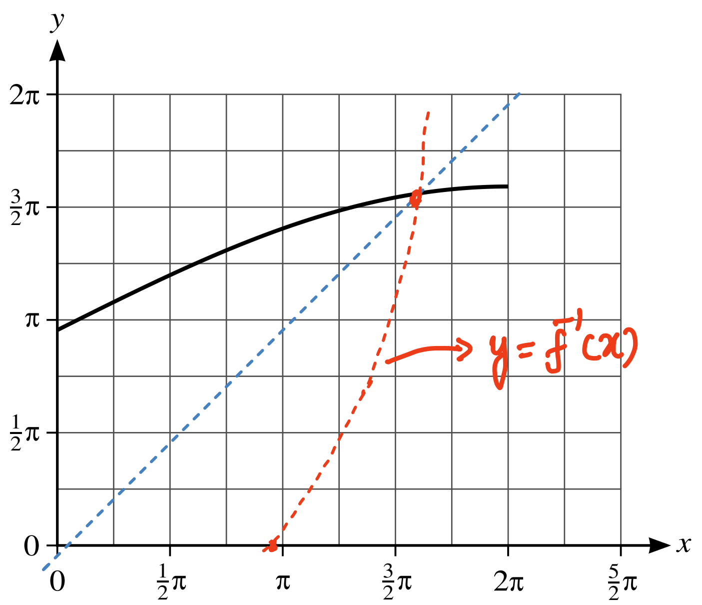
$\theta=-\pi/6$





$$\left(\theta=0,\ \frac{\pi}{6},\ \frac{\pi}{2},\ \frac{5\pi}{6}\right)$$

5.



The diagram shows the graph of $y = \mathrm{f}(x)$ where the function f is defined by

$$\mathrm{f}(x) = 3 + 2\sin\tfrac{1}{4}x \text{ for } 0 \leqslant x \leqslant 2\pi.$$

(a) On the diagram above, sketch the graph of $y = \mathrm{f}^{-1}(x)$. [2]

Reflect along $y=x$

(b) Find an expression for $\mathrm{f}^{-1}(x)$. [2]

$$y = 3 + 2\sin\left(\frac{1}{4}x\right)$$

$$y - 3 = 2\sin\left(\frac{1}{4}x\right)$$

$$\frac{y-3}{2} = \sin\left(\frac{1}{4}x\right)$$

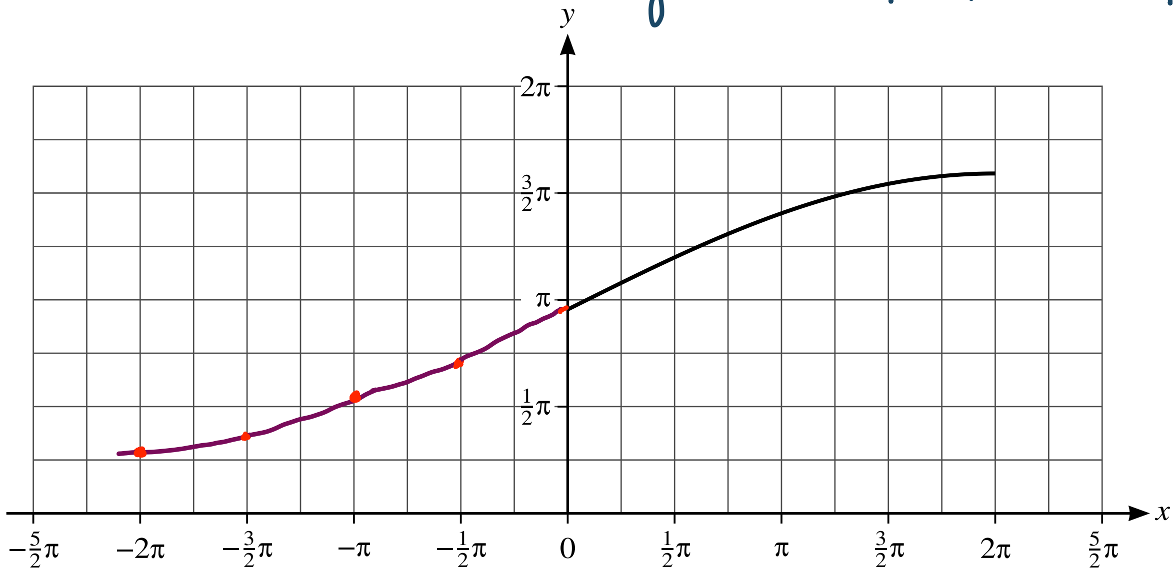
$$\frac{1}{4}x = \sin^{-1}\left(\frac{y-3}{2}\right)$$

$$x = 4\sin^{-1}\left(\frac{y-3}{2}\right)$$

$$f^{-1}(x) = 4\sin^{-1}\left(\frac{x-3}{2}\right)$$

x	$-\frac{1}{2}\pi$	$-\pi$	$-\frac{3}{2}\pi$	-2π
$y = g(x)$	2.23	1.59	1.15	1

(c)



The diagram above shows part of the graph of the function $g(x) = 3 + 2\sin\frac{1}{4}x$ for $-2\pi \leqslant x \leqslant 2\pi$.

Complete the sketch of the graph of $g(x)$ on the diagram above and hence explain whether the function g has an inverse. [2]

Yes, the function g has inverse because it is an one to one function.

(d) Describe fully a sequence of three transformations which can be combined to transform the graph of $y = \sin x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$ to the graph of $y = f(x)$, making clear the order in which the transformations are applied. [6]

$y = \sin x$

$y = \sin\left(\frac{x}{4}\right)$ Stretch by factor 4 in x direction

$y = 2\sin\left(\frac{x}{4}\right)$ Stretch by factor 2 in y-direction

$y = 3 + 2\sin\left(\frac{x}{4}\right)$ Translation by vector $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

6.

The equation of a circle is $(x-a)^2 + (y-3)^2 = 20$. The line $y = \frac{1}{2}x + 6$ is a tangent to the circle at the point P.

(a) Show that one possible value of a is 4 and find the other possible value. [5]

For intersection

$$(x-a)^2 + \left(\frac{x}{2}+6-3\right)^2 = 20$$

$(a+b)^2 = a^2+2ab+b^2$

$$x^2 - 2ax + a^2 + \left(\frac{x}{2}+3\right)^2 = 20$$

$$x^2 - 2ax + a^2 + \frac{x^2}{4} + 3x + 9 = 20$$

(multiply by 4)

$$4x^2 - 8ax + 4a^2 + x^2 + 12x + 36 = 80$$

$$5x^2 + 12x - 8ax + 4a^2 + 36 - 80 = 0$$

$$\underbrace{5x^2}_{a} + \underbrace{(12-8a)}_{b}x + \underbrace{(4a^2-44)}_{c} = 0$$

For tangency:

$$b^2 - 4ac = 0$$

$$(12-8a)^2 - 4 \times 5 \times (4a^2-44) = 0$$

$$144 - 192a + 64a^2 - 20(4a^2-44) = 0$$

$$144 - 192a + 64a^2 - 80a^2 + 880 = 0$$

$$-16a^2 - 192a + 1024 = 0$$

Divide by -16 on both sides

$$a^2 + 12a - 64 = 0$$

$$(a-4)(a+16) = 0$$

$a - 4 = 0$ $\quad$ $a + 16 = 0$

$a = 4$ $\quad$ $a = -16$

(b) For $a = 4$, find the equation of the normal to the circle at P. [4]

$a = 4$, $5x^2 + (12 - 8a)x + (4a^2 - 44) = 0$

$5x^2 + (12 - 8\times4)x + (4\times4^2 - 44) = 0$

$5x^2 - 20x + 20 = 0$

Divide 5 on both sides

$x^2 - 4x + 4 = 0$

$(x-2)(x-2) = 0$

$x - 2 = 0$

$x = 2$

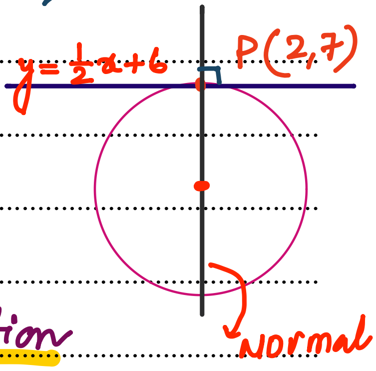
when $x = 2$,

$y = \frac{1}{2}(2) + 6$

$y = 7$

Point of intersection

$(2, 7)$



Gradient of Normal

$m = \frac{-1}{1/2} = -2$

Equation of Normal

$y - y_1 = m(x - x_1)$

$y - 7 = -2(x - 2)$

$y - 7 = -2x + 4$

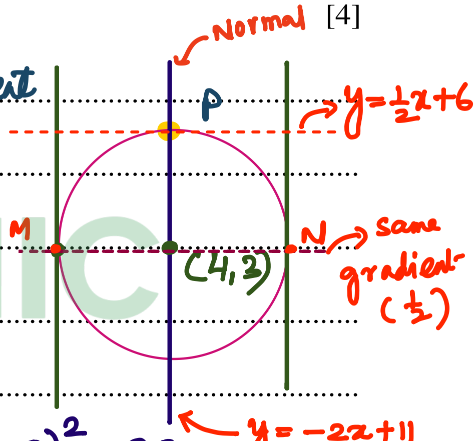
$y = -2x + 11$

(c) For $a = 4$, find the equations of the two tangents to the circle which are parallel to the normal found in **(b)**. [4]

Parallel lines have the same gradient

Circle equation, $(x-4)^2 + (y-3)^2 = 20$

Given $a = 4$, Centre of Circle $(4, 3)$



Find the equation of line MN

Gradient $= \frac{1}{2}$

$y - y_1 = m(x - x_1)$

$y - 3 = \frac{1}{2}(x - 4)$

$y = \frac{1}{2}x - 2$

$y = \frac{1}{2}x - 2 + 3$

$y = \frac{1}{2}x + 1$

Substituting the value of y in equation of Circle:

$(x-4)^2 + (\frac{1}{2}x + 1 - 3)^2 = 20$

$(x-4)^2 + (\frac{1}{2}x - 2)^2 = 20$

$x^2 - 8x + 16 + \frac{1}{4}x^2 - 2x + 4 = 20$

$\frac{5}{4}x^2 - 10x + 20 = 20$

$\frac{5}{4}x^2 - 10x = 0$

$5x^2 - 40x = 0$

$5x(x - 8) = 0$

$x = 0$ or $x = 8$

$x = 0$, $y = \frac{1}{2}(0) + 1 = 1$

$M(0, 1)$

$x = 8$, $y = \frac{1}{2}(8) + 1 = 5$

$N(8, 5)$

Tangent through M

$y - 1 = -2(x - 0)$

$y = -2x + 1$

Tangent through N

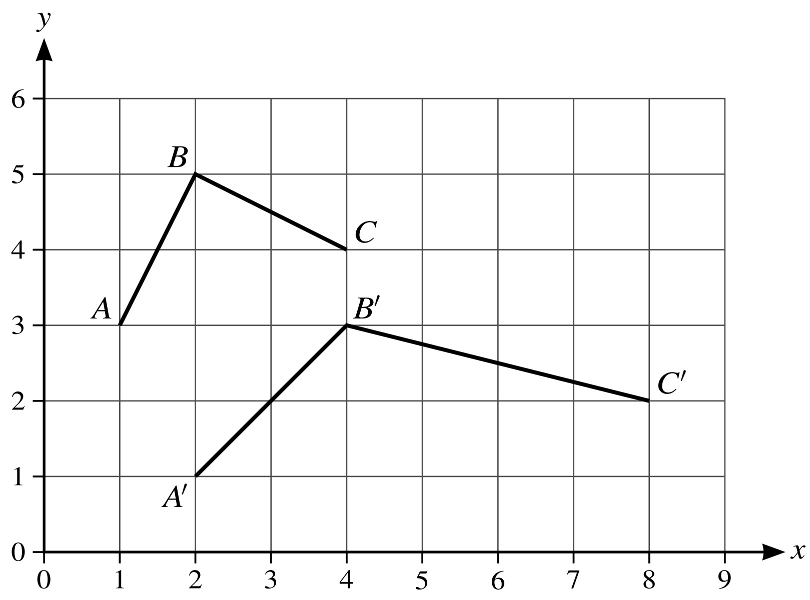
$y - 5 = -2(x - 8)$

$y - 5 = -2x + 16$

$y = -2x + 21$

MATHTONIC

7.



The diagram shows the graph of $y = \mathrm{f}(x)$, which consists of the two straight lines AB and BC. The lines $A'B'$ and $B'C'$ form the graph of $y = \mathrm{g}(x)$, which is the result of applying a sequence of two transformations, in either order, to $y = \mathrm{f}(x)$.

State fully the two transformations. [4]

$A(1, 3)$ $\quad B(2, 5)$ $\quad C(4, 4)$

$\times 2 \quad -2$

$A'(2, 1)$ $\quad B'(4, 3)$ $\quad C'(8, 2)$

x - Coordinates are multiplied by 2.

#① Stretch, Parallel to x-axis, Scale factor 2.

y - Coordinates are reduced by 2.

#② Translation by vector $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

8.

The function f is defined for $x \in \mathbb{R}$ by $f(x) = x^2 - 6x + c$, where c is a constant. It is given that $f(x) > 2$ for all values of x.

Find the set of possible values of c. [4]

$x^2 - 6x + c > 2$

$x^2 - 6x + c - 2 > 0$

Applying Completing Square

$(x-3)^2 - 9 + c - 2 > 0$

$(x-3)^2 + c - 11 > 0$

This is always positive (≥ 0), because it is squared.

The minimum value of $(x-3)^2$ is 0.

Then, $0 + c - 11 > 0$

$c - 11 > 0$

$c > 11$

MATH TONIC

9.

(a) Give the complete expansion of $\left(x+\frac{2}{x}\right)^5$. [2]

$$\left(x+\frac{2}{x}\right)^5 = x^5 + {}^5C_1 x^4\left(\frac{2}{x}\right) + {}^5C_2 x^3\left(\frac{2}{x}\right)^2 + {}^5C_3 x^2\left(\frac{2}{x}\right)^3 + {}^5C_4 x\left(\frac{2}{x}\right)^4 + {}^5C_5\left(\frac{2}{x}\right)^5$$

$$= x^5 + 10x^3 + 40x + \frac{80}{x} + \frac{80}{x^3} + \frac{32}{x^5}$$

(b) In the expansion of $(a+bx^2)\left(x+\frac{2}{x}\right)^5$, the coefficient of x is zero and the coefficient of $\frac{1}{x}$ is 80.

Find the values of the constants a and b. [4]

$$(a+bx^2)\left(x+\frac{2}{x}\right)^5 = (a+bx^2)\left(x^5+10x^3+40x+\frac{80}{x}+\frac{80}{x^3}+\frac{32}{x^5}\right)$$

$a \times 40x = 40ax$

$bx^2 \times \frac{80}{x} = 80bx$

Coefficient of $x = 0$,

$40a + 80b = 0$

$a + 2b = 0$ —(i)

$a \times \frac{80}{x} = 80a \times \frac{1}{x}$

$bx^2 \times \frac{80}{x^3} = 80b \times \frac{1}{x}$

Coefficient of $\frac{1}{x} = 80$

$80a + 80b = 80$

$a + b = 1$ —(ii)

$$\begin{aligned} a+2b&=0 \\ -\;(a+b&=1) \\ \hline b&=-1 \end{aligned}$$

$a + b = 1$

$a - 1 = 1$

$a = 2$

10.

(a) Show that the equation

$$3\tan^2 x - 3\sin^2 x - 4 = 0$$

may be expressed in the form $a\cos^4 x + b\cos^2 x + c = 0$, where a, b and c are constants to be found. [3]

$3\tan^2 x - 3\sin^2 x - 4 = 0$

$3\dfrac{\sin^2 x}{\cos^2 x} - 3\sin^2 x - 4 = 0$

$3\sin^2 x - 3\sin^2 x \cdot \cos^2 x - 4\cos^2 x = 0$

$3(1-\cos^2 x) - 3(1-\cos^2 x)\cdot\cos^2 x - 4\cos^2 x = 0$

$3 - 3\cos^2 x - 3(\cos^2 x - \cos^4 x) - 4\cos^2 x = 0$

$3 - 3\cos^2 x - 3\cos^2 x + 3\cos^4 x - 4\cos^2 x = 0$

$3\cos^4 x - 10\cos^2 x + 3 = 0$

MATH TONIC

(b) Hence solve the equation $3\tan^2 x - 3\sin^2 x - 4 = 0$ for $0° \leqslant x \leqslant 180°$. [4]

$3\cos^4 x - 10\cos^2 x + 3 = 0$

Let $y = \cos^2 x$
$y^2 = \cos^4 x$

$3y^2 - 10y + 3 = 0$

$(3y-1)(y-3) = 0$

$3y - 1 = 0$ $\quad\quad$ $y - 3 = 0$

$3y = 1$ $\quad\quad$ $y = 3$

$y = \frac{1}{3}$ $\quad\quad$ $\cos^2 x = 3$

$\cos^2 x = \frac{1}{3}$ $\quad\quad$ $\cos x = \sqrt{3}$ or $\cos x = -\sqrt{3}$

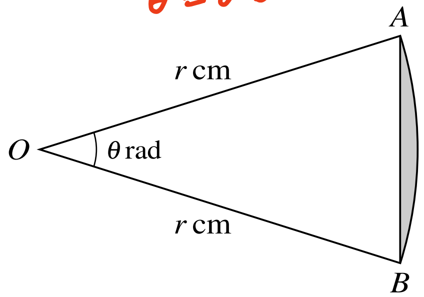
NOT POSSIBLE / NO SOLUTIONS

$\cos x = \sqrt{\frac{1}{3}}$ or $\cos x = -\sqrt{\frac{1}{3}}$

$x = 54.7°$ or $x = 125.3°$

11.

$r = 16\text{cm}$
$\theta = 0.6 \text{ rad.}$



The diagram shows a sector OAB of a circle with centre O and radius r cm. Angle $AOB = \theta$ radians. It is given that the length of the arc AB is 9.6 cm and that the area of the sector OAB is 76.8 cm^2.

(a) Find the area of the shaded region. [5]

Area of Sector $= 76.8$

$\frac{1}{2}r^2\theta = 76.8$

$\frac{1}{2}r^2 \times \left(\frac{9.6}{r}\right) = 76.8$

$4.8r = 76.8$

$r = \frac{76.8}{4.8}$

$r = 16$

Length of Arc $AB = 9.6$

$r\theta = 9.6$

$\theta = \frac{9.6}{r}$

$\theta = \frac{9.6}{16}$

$\theta = 0.6$ rad

Shaded Area $=$ Area of Sector OAB $-$ Area of $\Delta AOAB$

$= 76.8 - \frac{1}{2} \times 16 \times 16 \times \sin(0.6)$

$= 4.5257$

$= 4.53 \text{ cm}^2$ (3sf)

(b) Find the perimeter of the shaded region. [2]

Using Cosine Rule: $AB = \sqrt{16^2 + 16^2 - 2 \times 16 \times 16 \times \cos(0.6)}$

$AB = \sqrt{512 - 512\cos(0.6)}$

$AB = 9.46$

Perimeter of shaded region: Arc length AB + $\overline{AB}$

$9.6 + 9.46 = 19.1$ (3sf)

MATHTONIC

$\boxed{\frac{0}{x} = 0 \quad \frac{x}{0} = \infty}$

12.

The function f is defined by $f(x) = 2 - \frac{5}{x+2}$ for $x > -2$.

(a) State the range of f. [1]

Since we are Subtracting a term from 2, all values of f(x) must be less than 2.

So Range of $f(x) < 2$

Alternatively: Substitute Something a little bigger than -2.

$f(-1.999) = -\infty$

$f(9999) = 2$

Substitute a very big number.

from $-\infty$ to 2

(b) Obtain an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [4]

$y = 2 - \frac{5}{x+2}$

$y - 2 = \frac{-5}{x+2}$

$2 - y = \frac{5}{x+2}$

$(2-y)(x+2) = 5$

$2x + 4 - xy - 2y = 5$

$2x - xy = 5 - 4 + 2y$

$x(2-y) = 1 + 2y$

$x = \frac{1+2y}{2-y}$

$f^{-1}(x) = \frac{1+2x}{2-x}$

Domain of $f^{-1}(x)$ = Range of $f(x)$

Domain of $f^{-1}(x) = x < 2$

MATH TONIC

The function g is defined by g(x) = $x + 3$ for $x > 0$.

(c) Obtain an expression for fg(x) giving your answer in the form $\dfrac{ax+b}{cx+d}$, where a, b, c and d are integers. [3]

$$f(x) = 2 - \frac{5}{x+2}$$

$$fg(x) = 2 - \frac{5}{g(x)+2}$$

$$fg(x) = 2 - \frac{5}{x+3+2}$$

$$= 2 - \frac{5}{x+5}$$

$$= \frac{2(x+5)-5}{x+5}$$

$$= \frac{2x+10-5}{x+5}$$

$$= \frac{2x+5}{x+5}$$

13.

The coefficient of x^3 in the expansion of $(3 + 2ax)^5$ is six times the coefficient of x^2 in the expansion of $(2 + ax)^6$.

Find the value of the constant a. [4]

$$(3+2ax)^5 = 3^5 + {}^5C_1 3^4(2ax) + {}^5C_2 3^3(2ax)^2 + \underbrace{{}^5C_3 3^2(2ax)^3}$$

$$90 \times 8a^3x^3$$

Coefficient of x^3 : $720a^3$

$$(2+ax)^6 = 2^6 + {}^6C_1 2^5(ax) + \underbrace{{}^6C_2 2^4(ax)^2}$$

$$240a^2x^2$$

Coefficient of x^2 : $240a^2$

Coeffient of x^3 = $6 \times$ Coefficient of x^2

$$720a^3 = 6 \times 240a^2$$

$$720a^3 = 1440a^2$$

$$a^3 = \frac{1440}{720}a^2$$

$$a^3 = 2a^2$$

$$a^3 - 2a^2 = 0$$

$$a^2(a-2) = 0$$

$a^2 = 0$ $\qquad$ $a - 2 = 0$

$a = 0$ $\qquad$ $a = 2$

NOT POSSIBLE

MATH TONIC

14.

(a) Verify the identity $(2x-1)(4x^2+2x-1) \equiv 8x^3-4x+1$. [1]

$8x^3 + \cancel{4x^2} - 2x - \cancel{4x^2} - 2x + 1$

$8x^3 - 4x + 1$

(b) Prove the identity $\dfrac{\tan^2\theta+1}{\tan^2\theta-1} \equiv \dfrac{1}{1-2\cos^2\theta}$. [3]

$$= \frac{\dfrac{\sin^2\theta}{\cos^2\theta}+1}{\dfrac{\sin^2\theta}{\cos^2\theta}-1}$$

$$= \frac{\dfrac{\sin^2\theta+\cos^2\theta}{\cancel{\cos^2\theta}}}{\dfrac{\sin^2\theta-\cos^2\theta}{\cancel{\cos^2\theta}}}$$

$$= \frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta-\cos^2\theta}$$

$$= \frac{1}{(1-\cos^2\theta)-\cos\theta}$$

$$= \frac{1}{1-2\cos^2\theta}$$

MATH TONIC

(c) Using the results of **(a)** and **(b)**, solve the equation

$$\frac{\tan^2\theta + 1}{\tan^2\theta - 1} = 4\cos\theta,$$

for $0° \leqslant \theta \leqslant 180°$. [5]

$$\frac{1}{1-2\cos^2\theta} = 4\cos\theta$$

$$1 = 4\cos\theta(1-2\cos^2\theta)$$

$$1 = 4\cos\theta - 8\cos^3\theta$$

$$8\cos^3\theta - 4\cos\theta + 1 = 0$$

let $x = \cos\theta$

$$8x^3 - 4x + 1 = 0$$

From part (a)

$$(2x-1)(4x^2+2x-1) = 0$$

$2x - 1 = 0$ $\qquad$ $4x^2 + 2x - 1 = 0$

$x = \frac{1}{2}$ $\qquad$ $x = \frac{-2 \pm \sqrt{2^2 - 4(4)(-1)}}{2(4)}$

$x = \frac{1}{2}$	$x = \frac{-1+\sqrt{5}}{4}$	$x = \frac{-1-\sqrt{5}}{4}$
$\cos\theta = \frac{1}{2}$	$\cos\theta = \frac{-1+\sqrt{5}}{4}$	$\cos\theta = \frac{-1-\sqrt{5}}{4}$
$\theta = \cos^{-1}\left(\frac{1}{2}\right)$	$\theta = \cos^{-1}\left(\frac{-1+\sqrt{5}}{4}\right)$	$\theta = \cos^{-1}\left(\frac{-1-\sqrt{5}}{4}\right)$
$\theta = 60°$	$\theta = 72°$	$\theta = 144°$

$$\theta = 60°, 72°, 144°$$

15.

Functions f and g are defined by

$$f(x) = (x + a)^2 - a \text{ for } x \leqslant -a,$$
$$g(x) = 2x - 1 \text{ for } x \in \mathbb{R},$$

where a is a positive constant.

(a) Find an expression for $f^{-1}(x)$. [3]

$y = (x+a)^2 - a$

$y + a = (x+a)^2$

$x + a = \pm\sqrt{y+a}$

$x + a = -\sqrt{y+a}$

$x = -a - \sqrt{y+a}$

$f^{-1}(x) = -a - \sqrt{x+a}$

Given that

$x \leq -a$

$\underbrace{x + a}_{\text{always Negative}} \leq 0$

we will consider negative square root.

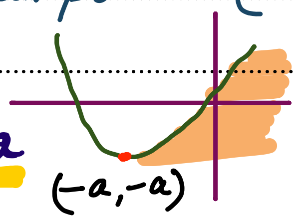
MATH TONIC

(b) **(i)** State the domain of the function f^{-1}. [1]

$f(x) = (x+a)^2 - a$, Minimum point: $(-a, -a)$

Domain of $f^{-1}(x)$ = Range of $f(x)$

$x \geqslant -a$



(ii) State the range of the function f^{-1}. [1]

Range of $f^{-1}(x)$ = Domain of $f(x)$ $f^{-1}(x) \leq -a$

(c) Given that $a = \frac{7}{2}$, solve the equation $gf(x) = 0$. [3]

$f(x) = (x+a)^2 - a$

$f(x) = \left(x + \frac{7}{2}\right)^2 - \frac{7}{2}$

$g(x) = 2x - 1$

$gf(x) = 2f(x) - 1$

$gf(x) = 2\left[\left(x + \frac{7}{2}\right)^2 - \frac{7}{2}\right] - 1$

$= 2\left(x + \frac{7}{2}\right)^2 - 7 - 1$

$= 2\left(x + \frac{7}{2}\right)^2 - 8$

$gf(x) = 0$

$2\left(x + \frac{7}{2}\right)^2 - 8 = 0$

$2\left(x + \frac{7}{2}\right)^2 = 8$

$\left(x + \frac{7}{2}\right)^2 = 4$

$x + \frac{7}{2} = \pm 2$

$x = 2 - \frac{7}{2}$ or $x = -2 - \frac{7}{2}$

$x = \frac{-3}{2}$ or $x = \frac{-11}{2}$

This is outside of given domain

MATH TONIC

16.

The coordinates of points A, B and C are $(6, 4)$, $(p, 7)$ and $(14, 18)$ respectively, where p is a constant. The line AB is perpendicular to the line BC.

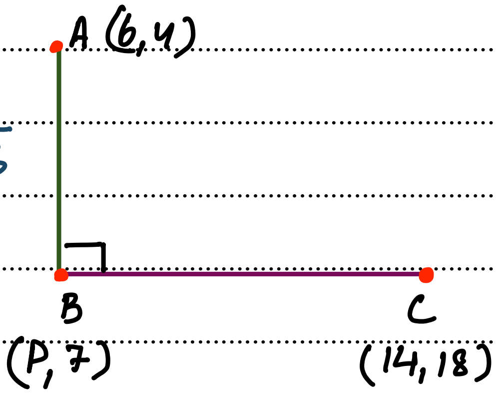
(a) Given that $p < 10$, find the value of p. [4]

Gradient of AB :

Formula

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$$\frac{7-4}{P-6} = \frac{3}{P-6}$$



Gradient of BC :

$$\frac{18-7}{14-P} = \frac{11}{14-P}$$

AB and BC are perpendicular :

So, $\text{Gradient}_{AB} \times \text{Gradient}_{BC} = -1$

$$\frac{3}{P-6} \times \frac{11}{14-P} = -1$$

$$\frac{33}{(P-6)(14-P)} = -1$$

$$33 = -(P-6)(14-P)$$

$$33 = -(14P - P^2 - 84 + 6P)$$

$$33 = -14P + P^2 + 84 - 6P$$

$$33 = P^2 - 20P + 84$$

$$P^2 - 20P + 51 = 0$$

$$(P-3)(P-17) = 0$$

$P - 3 = 0$ or $P - 17 = 0$

$P = 3$ $\qquad$ $P = 17$ → NOT POSSIBLE

(Give that $P < 10$)

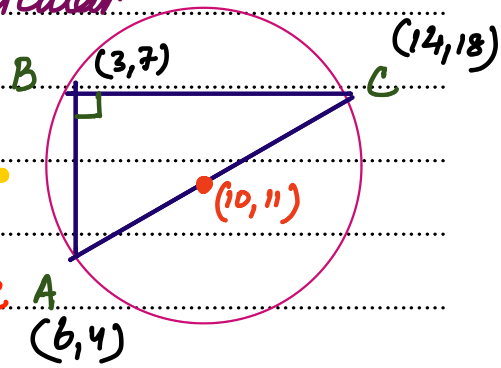
MATH TONIC

A circle passes through the points A, B and C.

(b) Find the equation of the circle. [3]

Since AB and BC are perpendicular then AC is the diameter.

Angle in Semicircle is 90° (Circle theorem)



Centre of Circle = Midpoint of AC

Centre $\left[\frac{6+14}{2}, \frac{4+18}{2}\right]$

Centre $(10, 11)$

Equation of Circle: $(x-10)^2+(y-11)^2=r^2$

Substitute any Coordinate: A, B or C

$(6-10)^2+(4-11)^2=r^2$

$(-4)^2+(-7)^2=r^2$

$16+49=r^2$

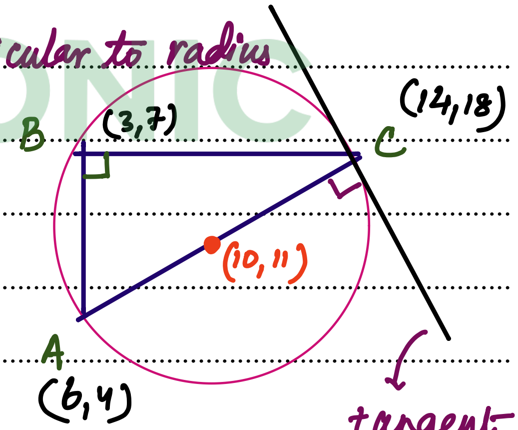
$r^2=65$

Here A Coordinate are Substituted

$(x-10)^2+(y-11)^2=65$

(c) Find the equation of the tangent to the circle at C, giving the answer in the form $dx+ey+f=0$, where d, e and f are integers. [3]

A tangent is always perpendicular to radius



Gradient of Radius:

$\frac{18-11}{14-10}=\frac{7}{4}$

$\text{Gradient}_{\text{Tangent}} \times \text{Gradient}_{\text{Radius}} = -1$

$m_{\text{tangent}} = \frac{-1}{7/4} = \frac{-4}{7}$

Equation of tangent:

$y-y_1=m(x-x_1)$

$y-18=\frac{-4}{7}(x-14)$

$7y-126=-4(x-14)$

$7y-126=-4x+56$

$4x+7y-126-56=0$

$4x+7y-182=0$

17.

A line has equation $y = 6x - c$ and a curve has equation $y = cx^2 + 2x - 3$, where c is a constant. The line is a tangent to the curve at point P.

Find the possible values of c and the corresponding coordinates of P. [7]

$y = 6x - c \qquad y = cx^2 + 2x - 3$

Solving Simultaneous Equations

$$cx^2 + 2x - 3 = 6x - c$$
$$cx^2 + 2x - 6x + c - 3 = 0$$
$$cx^2 - 4x + (c-3) = 0$$

For tangency:

$$b^2 - 4ac = 0$$
$$(-4)^2 - 4c(c-3) = 0$$
$$16 - 4c^2 + 12c = 0$$
$$-4c^2 + 12c + 16 = 0$$

(Divide by -4) $\qquad c^2 - 3x - 4 = 0$

$$(c-4)(c+1) = 0$$

$c - 4 = 0$

$c = 4$

$cx^2 - 4x + (c-3) = 0$

$4x^2 - 4x + (4-3) = 0$

$4x^2 - 4x + 1 = 0$

$(2x-1)(2x-1) = 0$

$2x - 1 = 0$

$x = \frac{1}{2}$

$y = 6x - 4$

$y = 6(\frac{1}{2}) - 4$

$y = -1$

when $c = 4$, $P(\frac{1}{2}, -1)$

$c + 1 = 0$

$c = -1$

$cx^2 - 4x + (c-3) = 0$

$-x^2 - 4x + (-1-3) = 0$

$-x^2 - 4x - 4 = 0$

$x^2 + 4x + 4 = 0$

$(x+2)(x+2) = 0$

$x + 2 = 0$

$x = -2$

$y = 6(-2) + 1$

$y = -11$

when $c = -1$, $P(-2, -11)$

MATH TONIC

18.

$\frac{1}{\infty} = 0$

$\frac{1}{0} = \infty$

The function f is defined by $f(x) = 1 + \dfrac{3}{x-2}$ for $x > 2$.

(a) State the range of f. [1]

Here domain is positive and $f(x)$ is always Calculate by adding values to 1. So, Range of $f(x) > 1$

Alternative:

$f(3) = 1 + \frac{3}{3-2} = 4$

$f(4) = 1 + \frac{3}{4-2} = 2.5$

$f(\infty) = 1 + \frac{3}{\infty} = 1$

$\frac{3}{\infty} \to 0$

Range > 1

(b) Obtain an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [4]

$y = 1 + \frac{3}{x-2}$

$y - 1 = \frac{3}{x-2}$

$(y-1)(x-2) = 3$

$xy - 2y - x + 2 = 3$

$xy - x = 3 - 2 + 2y$

$x(y-1) = 1 + 2y$

$x = \frac{1+2y}{y-1}$

$f^{-1}(x) = \frac{1+2x}{x-1}$

Domain of $f^{-1}(x)$ = Range of $f(x)$

Domain of $f^{-1}(x)$: $x > 1$

MATH TONIC

The function g is defined by $g(x) = 2x - 2$ for $x > 0$.

(c) Obtain a simplified expression for $gf(x)$. [2]

$g(x) = 2x - 2$

$g f(x) = 2f(x) - 2$

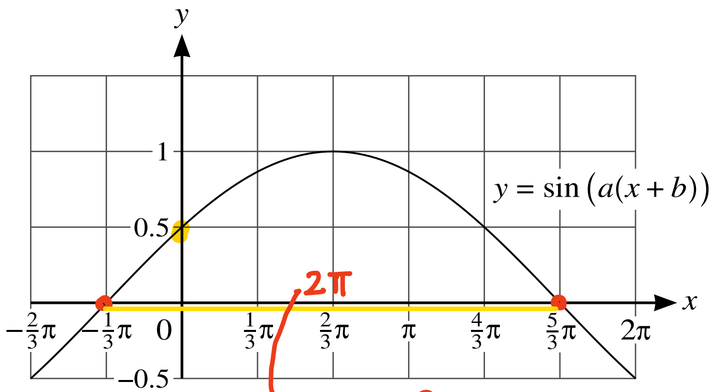
$gf(x) = 2\left[1 + \frac{3}{x-2}\right] - 2$

$= \cancel{2} + \frac{6}{x-2} \cancel{-2}$

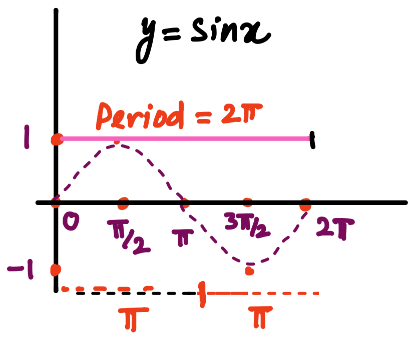
$gf(x) = \frac{6}{x-2}$

19.

Rule : $P = \frac{2\pi}{a}$



Half Period



The diagram shows part of the graph of $y = \sin\big(a(x+b)\big)$, where a and b are positive constants.

(a) State the value of a and one possible value of b. [2]

Distance between $-\frac{1}{3}\pi$ and $\frac{5}{3}\pi$:

$\frac{5}{3}\pi - \left(-\frac{1}{3}\pi\right) = \frac{5}{3}\pi + \frac{1}{3}\pi = 2\pi$

So the Period (P) $= 2 \times 2\pi = 4\pi$

$P = \frac{2\pi}{a}$

$a = \frac{2\pi}{P} = \frac{2\pi}{4\pi} = \frac{1}{2}$

$y = \sin\left(\frac{1}{2}(x+b)\right)$ Here $(0, 0.5)$

Substitute any coordinate lying on the curve

$0.5 = \sin\left(\frac{1}{2}b\right)$

$\frac{1}{2}b = \sin^{-1}(0.5)$

$\frac{1}{2}b = \frac{\pi}{6}$

$b = \frac{\pi}{3}$

$y = \sin\left(\frac{1}{2}\left(x + \frac{\pi}{3}\right)\right)$

Another curve, with equation $y = f(x)$, has a single stationary point at the point (p, q), where p and q are constants. This curve is transformed to a curve with equation

$$y = -3f\left(\tfrac{1}{4}(x+8)\right).$$

MATH TONIC

(b) For the transformed curve, find the coordinates of the stationary point, giving your answer in terms of p and q. [3]

Order of transformation :

$y_1 = f\left(\frac{1}{4}x\right)$ stretch by factor 4 parallel to x axis.

$(P, q) \longrightarrow (4P, q)$

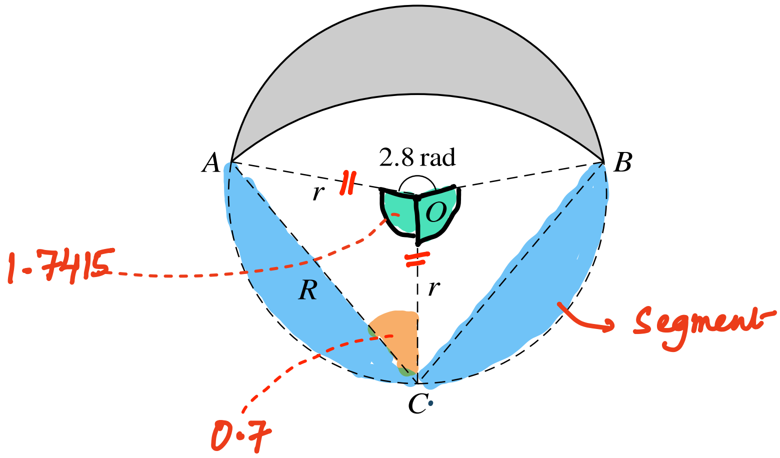
$y_2 = f\left(\frac{1}{4}(x+8)\right)$ Transformation by vector $\begin{pmatrix} -8 \\ 0 \end{pmatrix}$

$(4P, q) \longrightarrow (4P-8, q)$

$y_3 = -3f\left(\frac{1}{4}(x+8)\right)$ stretch by factor -3 parallel to y axis.

$(4P-8, q) \longrightarrow (4P-8, -3q)$

20.



The diagram shows points A, B and C lying on a circle with centre O and radius r. Angle AOB is 2.8 radians. The shaded region is bounded by two arcs. The upper arc is part of the circle with centre O and radius r. The lower arc is part of a circle with centre C and radius R.

(a) State the size of angle ACO in radians. [1]

In triangle AOC (Isosceles triangle):

$$\text{Angle } AOC = \frac{2\pi - 2.8}{2} = 1.7415$$

$$\text{Angle } ACO = \frac{\pi - 1.7415}{2} = 0.7$$

(b) Find R in terms of r. [1]

Using Sine Rule in triangle AOC

$$\frac{R}{\sin(1.7415)} = \frac{r}{\sin(0.7)}$$

$$R = \frac{r \times \sin(1.7415)}{\sin 0.7}$$

$$R = 1.53r$$

Alternatively: Using Cosine Rule

$$R^2 = r^2 + r^2 - 2r^2 \cos(1.7415)$$

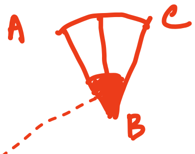
$$R^2 = 2r^2 - 2r^2 \cos(1.7415)$$

$$R^2 = 2r^2(1 - \cos(1.7415))$$

$$R = \sqrt{2r^2(1 - \cos(1.7415))}$$

$$R = 1.53r$$

MATH TONIC



(c) Find the area of the shaded region in terms of r. [7]

Shaded Area = Area of Circle − Area of Sector CAB − 2× Area of Segment AC or BC

$$= \pi r^2 - \frac{1}{2}R^2 \times 1.4 - 2\left[\frac{1}{2}\times r^2 \times 1.7415 - \frac{1}{2}r^2 \sin 1.7415\right]$$

$$= \pi r^2 - 0.7R^2 - r^2 \times 1.7415 - r^2 \sin(1.7415)$$

$$= \pi r^2 - 0.7(1.53r)^2 - r^2 \times 1.7415 - r^2 \sin(1.7415)$$

$$= r^2\left[\pi - 0.7 \times 1.53^2 - 1.7415 - \sin(1.7415)\right]$$

$$= 0.747r^2 \ (3sf)$$

MATH TONIC