1.

Expand $(3+x)(1-2x)^{\frac{1}{2}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

$$(1-2x)^{\frac{1}{2}} = 1 + \frac{1}{2}(-2x) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-2x)^2 + \ldots$$

$$= 1 - x + \frac{\frac{1}{2}\times -\frac{1}{2}}{2}(4x^2) + \ldots$$

$$= 1 - x + \left(\frac{-1}{8}\right)4x^2$$

$$= \underline{1 - x - \frac{1}{2}x^2}$$

formula

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$$

where n is rational and $|x| < 1$

$$(3+x)(1-2x)^{\frac{1}{2}} = (3+x)\left(1 - x - \frac{1}{2}x^2\right)$$

$$= 3 - 3x - \frac{3}{2}x^2 + x - x^2 - \frac{1}{2}x^3$$

$$= \underline{3 - 2x - \frac{5}{2}x^2}$$

(Ignore higher power of x)

2.

Solve the equation $\ln(x-5) = 7 - \ln x$. Give your answer correct to 2 decimal places. [4]

$\ln(x-5) = 7 - \ln x$

$\ln(x-5) + \ln x = 7$

$\ln x(x-5) = 7$

$x(x-5) = e^7$

$x^2 - 5x - e^7 = 0$

Rule

$a^x = b$

$\log_a b = x$

$x = \dfrac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-e^7)}}{2 \times 1}$

$x = 35.7096\ldots$

$x = 35.71$

Negative value of x is ignored because in $\ln x \rightarrow x > 0$

The equation of a curve is $y = \dfrac{e^{\sin x}}{\cos^2 x}$ for $0 \leqslant x \leqslant 2\pi$.

Find $\dfrac{dy}{dx}$ and hence find the x-coordinates of the stationary points of the curve. [7]

$$y = \frac{e^{\sin x}}{\cos^2 x}$$

$$\frac{dy}{dx} = \frac{\cos^2 x \cdot \frac{d}{dx}(e^{\sin x}) - e^{\sin x} \cdot \frac{d}{dx}(\cos^2 x)}{(\cos^2 x)^2}$$

$$\frac{dy}{dx} = \frac{\cos^2 x \cdot \cos x \cdot e^{\sin x} - e^{\sin x} \cdot 2\cos x(-\sin x)}{\cos^4 x}$$

$$\frac{dy}{dx} = \frac{\cancel{\cos x} \cdot e^{\sin x}(\cos^2 x + 2\sin x)}{\cancel{\cos^4 x}}$$

$$\frac{dy}{dx} = \frac{e^{\sin x}(\cos^2 x + 2\sin x)}{\cos^3 x}$$

At stationary point, $\dfrac{dy}{dx} = 0$

$$\frac{e^{\sin x}(\cos^2 x + 2\sin x)}{\cos^3 x} = 0$$

$$e^{\sin x}(\cos^2 x + 2\sin x) = 0$$

$e^{\sin x} = 0$ (Not defined)

$$\cos^2 x + 2\sin x = 0$$

$$1 - \sin^2 x + 2\sin x = 0$$

$$\sin^2 x - 2\sin x - 1 = 0$$

$$\sin x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times (-1)}}{2 \times 1}$$

$$\sin x = \frac{2 \pm 2\sqrt{2}}{2}$$

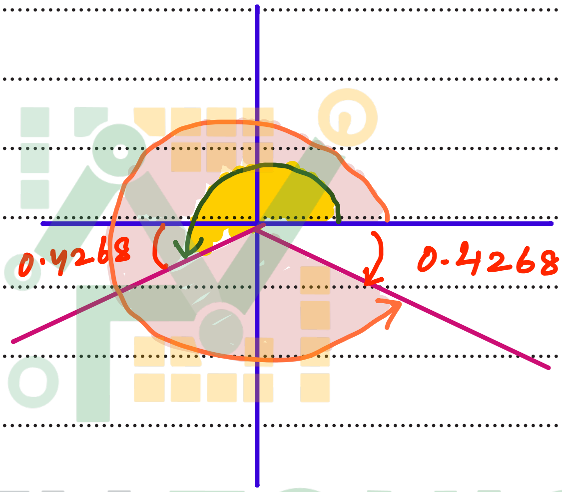
$$\sin x = 1 \pm \sqrt{2}$$

$\sin x = 1 + \sqrt{2}$
(NOT POSSIBLE)

$\sin x = 1 - \sqrt{2} = -0.414$

$x = \sin^{-1}(-0.414)$

$x = -0.4268$



$x = \pi + 0.4268 \quad , \quad 2\pi - 0.4268$

$x = 3.57 \quad , \quad 5.86$

4.

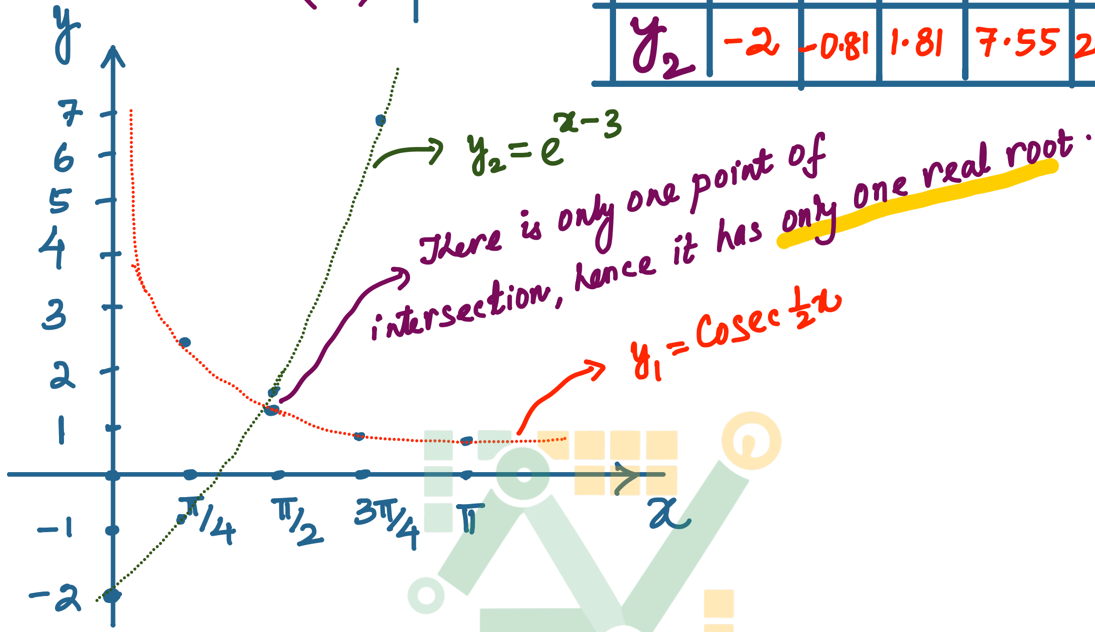
(a) By sketching a suitable pair of graphs, show that the equation $\operatorname{cosec}\frac{1}{2}x = e^x - 3$ has exactly one root, denoted by α, in the interval $0 < x < \pi$. [2]

$y_1 = \operatorname{cosec}\frac{1}{2}x$

$y_1 = \dfrac{1}{\sin(\frac{1}{2}x)}$

$y_2 = e^x - 3$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y_1	∞	2.61	1.41	1.08	1
y_2	-2	-0.81	1.81	7.55	20.14



(b) Verify by calculation that α lies between 1 and 2. [2]

$\dfrac{1}{\sin\frac{x}{2}} - e^x + 3 = 0$

let $f(x) = \dfrac{1}{\sin\frac{x}{2}} - e^x + 3$

$f(1) = \dfrac{1}{\sin(\frac{1}{2})} - e^1 + 3 = 2.367$

$f(2) = \dfrac{1}{\sin(1)} - e^2 + 3 = -3.2$

Since $f(1) > 0$ and $f(2) < 0$

There is a sign change

Hence α lies between 1 and 2.

(c) Show that if a sequence of values in the interval $0 < x < \pi$ given by the iterative formula

$$x_{n+1} = \ln\left(\operatorname{cosec}\tfrac{1}{2}x_n + 3\right)$$

converges, then it converges to α. [1]

Here we rearrange to get back the Original equation

$x = \ln\left(\operatorname{cosec}\frac{1}{2}x + 3\right)$

$e^x = \operatorname{cosec}\frac{1}{2}x + 3$

$e^x - 3 = \operatorname{cosec}\frac{1}{2}x$

(d) Use this iterative formula with an initial value of 1.4 to determine α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_{n+1} = \ln\left(\operatorname{cosec}\frac{1}{2}x_n + 3\right)$

Initial value $= 1.4$

$x_0 = 1.4$

$x_1 = \ln\left[\dfrac{1}{\sin\frac{1}{2}(1.4)} + 3\right] = 1.5156$

$x_2 = \ln\left[\dfrac{1}{\sin\frac{1}{2}(1.5156)} + 3\right] = 1.4940$

$x_3 = \ln\left[\dfrac{1}{\sin\frac{1}{2}(1.4940)} + 3\right] = 1.4978$

$x_4 = \ln\left[\dfrac{1}{\sin\frac{1}{2}(1.4978)} + 3\right] = 1.4971$

$x_5 = \ln\left[\dfrac{1}{\sin\frac{1}{2}(1.4971)} + 3\right] = 1.4972$

$\rightarrow 1.50$ (2dp)

$\alpha = 1.50$

(e) State the minimum number of calculated iterations needed with this initial value to determine α correct to 2 decimal places. [1]

Minimum number of Calculated iterations $= 4$ (four)

5.

$\boxed{\sin x = 1-u}$

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Use the substitution $u = 1 - \sin x$ to find the exact value of

$$\int_{\pi}^{\frac{3}{2}\pi} \frac{\sin 2x}{\sqrt{1-\sin x}}\,\mathrm{d}x.$$

Give your answer in the form $a + b\sqrt{2}$ where a and b are rational numbers to be determined. [7]

$u = 1 - \sin x$

$\frac{du}{dx} = -\cos x$

$du = -\cos x \cdot dx$

$dx = \frac{du}{-\cos x}$

$\sin 2x = 2\sin x \cdot \cos x$

$$\int_{\pi}^{\frac{3}{2}\pi} \frac{\sin 2x}{\sqrt{1-\sin x}} \cdot dx$$

$$\int_{\pi}^{\frac{3}{2}\pi} \frac{2\sin x \cdot \cos x}{\sqrt{1-\sin x}} \cdot dx$$

$$\int_{1}^{2} \frac{2(1-u)\cancel{\cos x}}{\sqrt{u}} \cdot \frac{du}{-\cancel{\cos x}}$$

Change of limits

$u = 1 - \sin x$

$x = \pi$, $u = 1 - \sin \pi = 1$

$x = \frac{3}{2}\pi$, $u = 1 - \sin\left(\frac{3\pi}{2}\right) = 2$

$$\int_{1}^{2} \frac{-2(1-u)}{\sqrt{u}} \cdot du$$

$$\int_{1}^{2} \frac{2u-2}{u^{1/2}} \cdot du$$

$$\int_{1}^{2} \left(\frac{2u}{u^{1/2}} - \frac{2}{u^{1/2}}\right) \cdot du$$

$$\int_{1}^{2} \left(2u^{1/2} - 2u^{-1/2}\right) \cdot du$$

$$2\int_{1}^{2} \left(u^{1/2} - u^{-1/2}\right) du$$

$$2\left[\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}\right]_1^2$$

$$2\left[\frac{2}{3}\times u^{3/2} - 2u^{1/2}\right]_1^2$$

$$2\left[\frac{2}{3}\times 2^{3/2} - \underline{2\times 2^{1/2}} - \left(\frac{2}{3}\times 1^{3/2} - 2\times 1^{1/2}\right)\right]$$

$$2\left[\underline{\frac{2}{3}\times 2^{3/2} - 2^{3/2}} - \underline{\frac{2}{3} + 2}\right]$$

$$2\left[2^{3/2}\left(\frac{2}{3} - 1\right) + \frac{4}{3}\right]$$

$$2\left[-\frac{1}{3}\times 2^{3/2} + \frac{4}{3}\right]$$

$$-\frac{2}{3}\times \underline{2^{3/2}} + \frac{8}{3}$$

$$-\frac{1}{3}\times 2\times 2\sqrt{2} + \frac{8}{3}$$

$$\frac{8}{3} - \frac{4}{3}\sqrt{2}$$

6.

(a) Given that $2x = \tan y$, show that $\frac{dy}{dx} = \frac{2}{1+4x^2}$. [3]

$2x = \tan y$

Differentiate w.r.t x

$2 = \sec^2 y \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{2}{\sec^2 y}$

$\frac{dy}{dx} = \frac{2}{1+\tan^2 y}$

$\frac{dy}{dx} = \frac{2}{1+(2x)^2}$

$\frac{dy}{dx} = \frac{2}{1+4x^2}$

(b) Hence find the exact value of $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} x \tan^{-1}(2x)\,dx$. [7]

$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \overbrace{x \underbrace{\tan^{-1}(2x)}_{u} \cdot dx}^{dv}$

I L A T E

Inverse trigonometric functions

Algebraic function

$u = \tan^{-1} 2x$

$2x = \tan u$

$\frac{du}{dx} = \frac{2}{1+4x^2}$

$du = \frac{2}{1+4x^2} \cdot dx$

From part (a)

$dv = x \cdot dx$

$\int dv = \int x \cdot dx$

$v = \frac{1}{2}x^2$

$u \cdot dv = uv - \int v \cdot du$

$= \frac{1}{2}x^2 \tan^{-1} 2x - \int \frac{1}{\cancel{2}} x^2 \times \frac{\cancel{2}}{1+4x^2} \cdot dx$

$$= \frac{1}{2}x^2\tan^{-1}2x - \int \frac{x^2}{1+4x^2} \cdot dx$$

Improper fraction

Degree of Numerator $\geq$ Degree of Denominator

for integration of improper fraction, use algebraic division to convert into a sum of whole number and proper fraction

$$\boxed{\frac{x^2}{1+4x^2} = \frac{1}{4} + \frac{-1/4}{1+4x^2}}$$

$$\begin{array}{r|l} & \frac{1}{4} \\ \hline 4x^2+1 & x^2 \\ & -(x^2 + 1/4) \\ \hline & -1/4 \end{array}$$

$$= \frac{1}{2}x^2\tan^{-1}(2x) - \int\left(\frac{1}{4} + \frac{-1/4}{1+4x^2}\right)dx$$

$$= \frac{1}{2}x^2\tan^{-1}(2x) - \left[\int \frac{1}{4} \cdot dx - \frac{1}{4}\int \frac{1}{1+4x^2} \cdot dx\right]$$

$$= \frac{1}{2}x^2\tan^{-1}(2x) - \frac{1}{4}x + \frac{1}{4}\int \frac{1}{4(x^2+1/4)} \cdot dx$$

$$= \frac{1}{2}x^2\tan^{-1}(2x) - \frac{1}{4}x + \frac{1}{4} \times \frac{1}{4}\int \frac{1}{x^2+\left(\frac{1}{2}\right)^2} \cdot dx$$

$$= \frac{1}{2}x^2\tan^{-1}(2x) - \frac{1}{4}x + \frac{1}{16} \times \frac{1}{1/2}\tan^{-1}\left(\frac{x}{1/2}\right)$$

$$= \frac{1}{2}x^2\tan^{-1}(2x) - \frac{1}{4}x + \frac{1}{8}\tan^{-1}(2x)$$

$$\int_{1/2}^{\sqrt{3}/2} x\tan^{-1}(2x) \cdot dx = \left[\frac{x^2}{2}\tan^{-1}(2x) - \frac{1}{4}x + \frac{1}{8}\tan^{-1}(2x)\right]_{1/2}^{\sqrt{3}/2}$$

$$= \frac{(\sqrt{3}/2)^2}{2}\tan^{-1}\left(2 \times \frac{\sqrt{3}}{2}\right) - \frac{1}{4} \times \frac{\sqrt{3}}{2} + \frac{1}{8}\tan^{-1}\left(2 \times \frac{\sqrt{3}}{2}\right)$$

$$- \left(\frac{(1/2)^2}{2}\tan^{-1}\left(2 \times \frac{1}{2}\right) - \frac{1}{4} \times \frac{1}{2} + \frac{1}{8}\tan^{-1}\left(2 \times \frac{1}{2}\right)\right)$$

$$= \frac{3}{8}\tan^{-1}\sqrt{3} - \frac{\sqrt{3}}{8} + \frac{1}{8}\tan^{-1}(\sqrt{3}) - \left(\frac{1}{8}\tan^{-1}(1) - \frac{1}{8} + \frac{1}{8}\tan^{-1}1\right)$$

$$= \frac{3}{8}\times\frac{\pi}{3} - \frac{\sqrt{3}}{8} + \frac{1}{8}\times\frac{\pi}{3} - \left(\frac{1}{8}\times\frac{\pi}{4} - \frac{1}{8} + \frac{1}{8}\times\frac{\pi}{4}\right)$$

$$= \frac{\pi}{8} - \frac{\sqrt{3}}{8} + \frac{\pi}{24} - \frac{\pi}{32} + \frac{1}{8} - \frac{\pi}{32}$$

$$= \pi\left(\frac{1}{8} + \frac{1}{24} - \frac{1}{32} - \frac{1}{32}\right) + \frac{1}{8} - \frac{\sqrt{3}}{8}$$

$$= \frac{5}{48}\pi + \frac{1}{8} - \frac{\sqrt{3}}{8}$$

7.

Express $\dfrac{6x^2-9x-16}{2x^2-5x-12}$ in partial fractions. [5]

when Degree of Numerator $\geq$ Degree of Denominator

Then it is an improper fraction.

First start long division.

$\dfrac{6x^2-9x-16}{2x^2-5x-12}$ Degree = 2, Degree = 2

Quotient

$$\begin{array}{r|l} & 3 \\ 2x^2-5x-12 & 6x^2-9x-16 \\ & 6x^2-15x-36 \\ & \quad\; - \;\; + \;\;\;\; + \\ \hline & \quad 6x+20 \end{array}$$

Remainder

$$\frac{6x^2-9x-16}{2x^2-5x-12} = 3 + \frac{6x+20}{2x^2-5x-12}$$

Factorise: $2x^2-5x-12$

$2x^2-8x+3x-12$

$2x(x-4)+3(x-4)$

$(2x+3)(x-4)$

Partial fraction

$$\frac{6x+20}{2x^2-5x-12} = \frac{6x+20}{(2x+3)(x-4)} = \frac{A}{2x+3} + \frac{B}{x-4}$$

$$\frac{6x+20}{(2x+3)(x-4)} = \frac{A(x-4)+B(2x+3)}{(2x+3)(x-4)}$$

$$6x+20 = A(x-4)+B(2x+3)$$

$x=4$: $6\times4+20 = 0 + B(2\times4+3)$

$44 = 11B$

$B = 4$

$x = -3/2$: $6(-3/2) + 20 = A(-3/2 - 4) + 0$

$$11 = -\frac{11}{2}A$$

$$A = -2$$

$$\frac{6x^2 - 9x - 16}{2x^2 - 5x - 12} = 3 + \frac{A}{2x+3} + \frac{B}{x-4}$$

$$\boxed{\frac{6x^2 - 9x - 16}{2x^2 - 5x - 12} = 3 - \frac{2}{2x+3} + \frac{4}{x-4}}$$

8. The variables x and y satisfy the equation $a^{2y-1} = b^{x-y}$, where a and b are constants.

(a) Show that the graph of y against x is a straight line. [3]

$a^{2y-1} = b^{x-y}$

$\ln a^{2y-1} = \ln b^{x-y}$

$(2y-1)\ln a = (x-y)\ln b$

$2y\ln a - \ln a = x\ln b - y\ln b$

$2y\ln a + y\ln b = x\ln b + \ln a$

$y(2\ln a + \ln b) = x\ln b + \ln a$

$y = \dfrac{x\ln b + \ln a}{2\ln a + \ln b}$

$y = \dfrac{x\ln b}{2\ln a + \ln b} + \dfrac{\ln a}{2\ln a + \ln b}$

$y = \dfrac{\ln b}{2\ln a + \ln b}x + \dfrac{\ln a}{2\ln a + \ln b}.$

like in straight line

$y = mx + c$

gradient   y-intercept

$y = \dfrac{\ln b}{2\ln a + \ln b}x + \dfrac{\ln a}{2\ln a + \ln b}.$

gradient   intercept

So it is a straight line

(b) Given that $a = b^3$, state the equation of the straight line in the form $y = px + q$, where p and q are rational numbers in their simplest form. [2]

$y = \dfrac{\ln b}{2\ln a + \ln b}x + \dfrac{\ln a}{2\ln a + \ln b}$

Given that $a = b^3$

$y = \dfrac{\ln b}{2\ln b^3 + \ln b}x + \dfrac{\ln b^3}{2\ln b^3 + \ln b}$

$y = \dfrac{\ln b}{\ln b^6 + \ln b}x + \dfrac{\ln b^3}{\ln b^6 + \ln b}$

$y = \dfrac{\ln b}{\ln(b^6 \times b)}x + \dfrac{\ln b^3}{\ln(b^6 \times b)}$

$$y = \frac{\ln b}{\ln b^7} x + \frac{\ln b^3}{\ln b^7}$$

$$y = \frac{\cancel{\ln b}}{7 \cancel{\ln b}} x + \frac{3 \cancel{\ln b}}{7 \cancel{\ln b}}$$

$$\boxed{y = \frac{1}{7} x + \frac{3}{7}}$$

logarithm formula applied:

$$\ln a + \ln b = \ln(ab)$$

$$\ln a^x = x \ln a$$

9.

The equation of a curve is $ye^{2x} + y^2e^x = 6$.

Find the gradient of the curve at the point where $y = 1$. [6]

for $y=1$

$$ye^{2x} + y^2e^x = 6$$
$$(1)e^{2x} + (1^2)e^x = 6$$
$$e^{2x} + e^x = 6$$
$$e^{2x} + e^x - 6 = 0$$
$$(e^x)^2 + e^x - 6 = 0$$

let $y = e^x$

$$y^2 + y - 6 = 0$$
$$(y+3)(y-2) = 0$$

$y = -3$ $\quad$ $y = 2$

$e^x = -3$ $\quad$ $\boxed{e^x = 2}$

NOT POSSIBLE

To find the gradient we need to find $\frac{dy}{dx}$

Differentiation:

Product Rule:

$$\boxed{\frac{d}{dx}(uv) = u\cdot v' + vu'}$$

$$ye^{2x} + y^2e^x = 6$$

$$\frac{d}{dx}(ye^{2x}) + \frac{d}{dx}(y^2e^x) = \frac{d}{dx}(6)$$

$$y\frac{d}{dx}(e^{2x}) + e^{2x}\frac{dy}{dx} + y^2\frac{d}{dx}(e^x) + e^x\cdot\frac{d}{dx}(y^2) = 0$$

$$y \times 2e^{2x} + e^{2x}\frac{dy}{dx} + y^2e^x + e^x \times 2y \times \frac{dy}{dx} = 0$$

$$2ye^{2x} + e^{2x}\frac{dy}{dx} + y^2e^x + 2ye^x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left[e^{2x} + 2ye^x\right] = -2ye^{2x} - y^2e^x$$

$$\frac{dy}{dx} = \frac{-2ye^{2x} - y^2e^x}{e^{2x} + 2ye^x}$$

$e^{2x} = e^x \times e^x$

$$\frac{dy}{dx} = \frac{-y\cancel{e^x}(2e^x + y)}{\cancel{e^x}(e^x + 2y)} = \frac{-y(2e^x + y)}{e^x + 2y}$$

For $y=1$, $e^x = 2$

$$\boxed{\frac{dy}{dx} = \frac{-1(2\times2+1)}{2+2(1)} = \frac{-5}{4}}$$

10.

(a) It is given that the equation $e^{2x} = 5 + \cos 3x$ has only one root.

Show by calculation that this root lies in the interval $0.7 < x < 0.8$. [2]

$e^{2x} = 5 + \cos 3x$

$e^{2x} - \cos 3x - 5 = 0$

$f(x) = e^{2x} - \cos 3x - 5$

$f(0.7) = e^{2(0.7)} - \cos 3(0.7) - 5 = -0.4399$

$f(0.8) = e^{2(0.8)} - \cos 3(0.8) - 5 = 0.6904$

Since there is a sign change for $f(0.7)$ and $f(0.8)$,

The root lies in the interval $0.7 < x < 0.8$

(b) Show that if a sequence of values in the interval $0.7 < x < 0.8$ given by the iterative formula

$$x_{n+1} = \frac{1}{2}\ln\left(5 + \cos 3x_n\right)$$

converges then it converges to the root of the equation in part **(a)**. [1]

Rearrange to get the Original equation

$x = \frac{1}{2}\ln(5 + \cos 3x)$

$2x = \ln(5 + \cos 3x)$

$\boxed{e^{2x} = 5 + \cos 3x}$

(c) Use this iterative formula to determine the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

$x_{n+1} = \frac{1}{2}\ln(5 + \cos 3x_n)$

Let the initial approximation be, $x_0 = \frac{0.7 + 0.8}{2} = 0.75$

$x_1 = \frac{1}{2}\ln(5 + \cos 3(0.75)) = 0.73759$

$x_2 = \frac{1}{2}\ln(5 + \cos 3(0.73759)) = 0.74094$

$x_3 = \frac{1}{2}\ln(5 + \cos 3(0.74094)) = 0.74003$

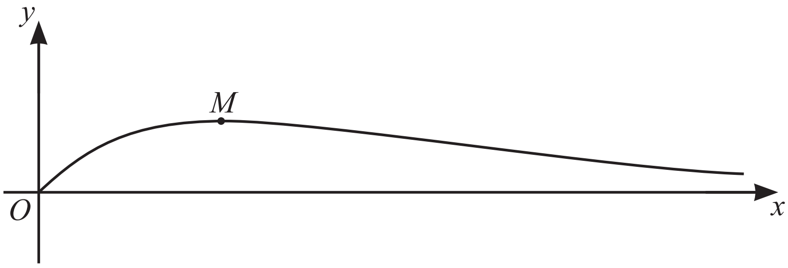
$x_4 = \frac{1}{2}\ln(5 + \cos 3(0.74003)) = 0.74028$

$x_5 = \frac{1}{2}\ln(5 + \cos 3(0.74028)) = 0.74021$

0.740 (3dp)

$x = 0.740$ (Correct to 3 dp)

11.



The diagram shows the curve $y = xe^{-ax}$, where a is a positive constant, and its maximum point M.

(a) Find the exact coordinates of M. [4]

At stationary point $\frac{dy}{dx} = 0$.

$y = xe^{-ax}$

$\frac{dy}{dx} = x \frac{d}{dx}(e^{-ax}) + e^{-ax} \cdot \frac{d}{dx}(x)$

$\frac{dy}{dx} = x(-a \cdot e^{-ax}) + e^{-ax}(1)$

$\frac{dy}{dx} = -axe^{-ax} + e^{-ax}$

Product Rule: $\boxed{\frac{d}{dx}(uv) = u \cdot v' + vu'}$

$0 = e^{-ax}(-ax+1)$

$e^{-ax} = 0$ NOT DEFINED

$-ax+1=0$

$-ax = -1$

$x = \frac{1}{a}$

$y = xe^{-ax}$

$y = \frac{1}{a} \times e^{-a \times \frac{1}{a}}$

$y = \frac{1}{a} e^{-1} = \frac{1}{ae}$

$\boxed{M\left(\frac{1}{a}, \frac{1}{ae}\right)}$

(b) Find the exact value of $\int_0^{\frac{2}{a}} xe^{-ax}\,dx$. [5]

I L A T E

Algebraic

Exponential

$$\int_0^{\frac{2}{a}} \underbrace{x}_{u}\, \overbrace{e^{-ax}\cdot dx}^{dv}$$

$$u = x \qquad dv = e^{-ax}\cdot dx$$

$$\frac{du}{dx} = 1 \qquad \int dv = \int e^{-ax}\cdot dx$$

$$du = dx \qquad v = \frac{e^{-ax}}{-a}$$

Type your text

Integration by Parts Rule:

$$u\cdot dv = uv - \int v\cdot du$$

Using Integration by parts:

$$\int x e^{-ax}\cdot dx = \frac{-x}{a}e^{-ax} - \int \frac{e^{-ax}}{-a}\cdot dx$$

$$= \frac{-x}{a}e^{-ax} + \frac{1}{a}\int e^{-ax}\cdot dx$$

$$= \frac{-x}{a}e^{-ax} + \frac{1}{a}\,\frac{e^{-ax}}{-a}$$

$$= \frac{-x}{a}e^{-ax} - \frac{e^{-ax}}{a^2}$$

$$\int_0^{\frac{2}{a}} x e^{-ax}\cdot dx = \left[-\frac{x}{a}e^{-ax} - \frac{e^{-ax}}{a^2}\right]_0^{2/a}$$

$$= \left(\frac{-\frac{2}{a}e^{-a(2/a)}}{a} - \frac{e^{-a(2/a)}}{a^2}\right) - \left(0 - \frac{e^0}{a^2}\right)$$

$$= \frac{-2e^{-2}}{a^2} - \frac{e^{-2}}{a^2} + \frac{1}{a^2}$$

$$= \frac{1}{a^2}\left[-2e^{-2} - e^{-2} + 1\right] = \boxed{\frac{1}{a^2}\left(1 - \frac{3}{e^2}\right)}$$

12.

(a) Show that $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$. [3]

$\cos^4\theta - \sin^4\theta$

$(\cos^2\theta)^2 - (\sin^2\theta)^2$

$(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)$

$1 \quad (\cos^2\theta - \sin^2\theta)$

$\cos 2\theta$

So, $\cos^4\theta - \sin^4\theta = \cos 2\theta$

$\boxed{\begin{array}{c} \text{Formula:} \\ \sin^2\theta + \cos^2\theta = 1 \\ a^2 - b^2 = (a+b)(a-b) \\ \cos 2\theta = \cos^2\theta - \sin^2\theta \end{array}}$

(b) Hence find the exact value of $\int_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi}\left(\cos^4\theta-\sin^4\theta+4\sin^2\theta\cos^2\theta\right)\mathrm{d}\theta$. [6]

$$\int_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi}(\underbrace{\cos^4\theta-\sin^4\theta}_{\text{from part(a)}}+4\sin^2\theta\cos^2\theta)\cdot d\theta$$

$$\int_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi}\left(\cos 2\theta+(2\sin\theta\cdot\cos\theta)^2\right)d\theta$$

$\boxed{\text{Formula: } \sin 2\theta = 2\sin\theta\cdot\cos\theta}$

$$\int_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi}\left(\cos 2\theta+(\sin 2\theta)^2\right)d\theta$$

$$\int_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi}\left(\cos 2\theta+\sin^2 2\theta\right)d\theta$$

$\cos 2\theta = 1-2\sin^2\theta$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\sin^2 2\theta = \frac{1-\cos 4\theta}{2}$

$$\int_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi}\left(\cos 2\theta+\frac{1-\cos 4\theta}{2}\right)d\theta$$

$$\int_{-\frac{1}{8}\pi}^{\frac{1}{8}\pi}\left(\cos 2\theta+\frac{1}{2}-\frac{\cos 4\theta}{2}\right)d\theta$$

$$\left[\frac{1}{2}\sin 2\theta+\frac{1}{2}\theta-\frac{1}{2}\times\frac{\sin 4\theta}{4}\right]_{-\frac{\pi}{8}}^{\frac{\pi}{8}}$$

$$\left(\frac{1}{2}\sin\frac{2\pi}{8}+\frac{1}{2}\times\frac{\pi}{8}-\frac{1}{8}\times\sin\left(\frac{4\pi}{8}\right)\right)-\left(\frac{1}{2}\sin\left(-\frac{2\pi}{8}\right)-\frac{\pi}{16}-\frac{1}{8}\sin\left(-\frac{4\pi}{8}\right)\right)$$

$$\left(\frac{1}{2}\times\frac{\sqrt{2}}{2}+\frac{\pi}{16}-\frac{1}{8}\right)-\left(\frac{1}{2}\times\left(\frac{-\sqrt{2}}{2}\right)-\frac{\pi}{16}+\frac{1}{8}\right)$$

$$\frac{\sqrt{2}}{4}+\frac{\pi}{16}-\frac{1}{8}+\frac{\sqrt{2}}{4}+\frac{\pi}{16}-\frac{1}{8}$$

$$\boxed{\frac{\sqrt{2}}{2}+\frac{\pi}{8}-\frac{1}{4}}$$

13.

Solve the equation $8^{3-6x} = 4 \times 5^{-2x}$. Give your answer correct to 3 decimal places. [4]

$$8^{3-6x} = 4 \times 5^{-2x}$$

$$\ln(8^{3-6x}) = \ln(4 \times 5^{-2x})$$

$$(3-6x)\ln 8 = \ln 4 + \ln 5^{-2x}$$

$$3\ln 8 - 6x\ln 8 = \ln 4 - 2x\ln 5$$

$$2x\ln 5 - 6x\ln 8 = \ln 4 - 3\ln 8$$

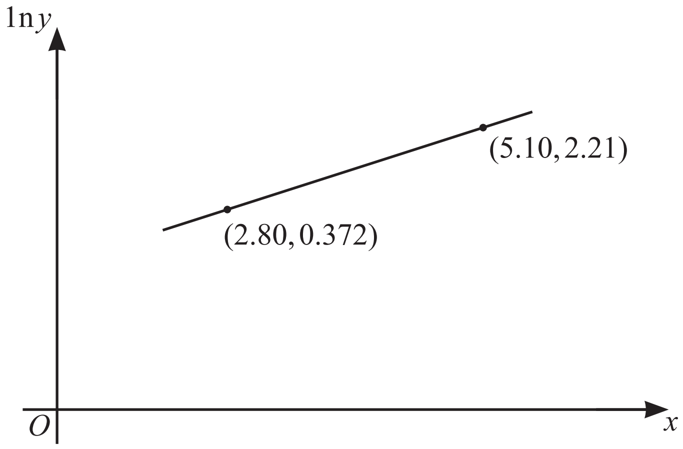
$$x(2\ln 5 - 6\ln 8) = \ln 4 - 3\ln 8$$

$$x = \frac{\ln 4 - 3\ln 8}{2\ln 5 - 6\ln 8}$$

$$x = 0.524$$

$$\boxed{\begin{array}{l}\text{logarithm formula applied:} \\ \ln a + \ln b = \ln(ab) \\ \ln a^x = x\ln a\end{array}}$$

14.



The variables x and y satisfy the equation $ky = e^{cx}$, where k and c are constants. The graph of $\ln y$ against x is a straight line passing through the points $(2.80, 0.372)$ and $(5.10, 2.21)$, as shown in the diagram.

Find the values of k and c. Give each value correct to 2 significant figures. [4]

$ky = e^{cx}$

$\ln(ky) = \ln(e^{cx})$

$\ln k + \ln y = cx \ln e$ $\quad \ln e = 1$

$\ln k + \ln y = cx \times 1$

$\ln k + \ln y = cx$

$\ln y = cx - \ln k$

The line is passing through two coordinates. Substitute Each.

Remember:
The graph is between x and $\ln y$ which means, coordinate $(x, \ln y)$

Substitute $(\underset{x}{2.8}, \underset{\ln y}{0.372})$

$\ln y = cx - \ln k$

$0.372 = 2.8c - \ln k$

$\ln k = 2.8c - 0.372$ —(i)

Substitute $(5.10, 2.21)$

$\ln y = cx - \ln k$

$2.21 = 5.1c - \ln k$

$\ln k = 5.1c - 2.21$ —(ii)

From equation (i) and (ii) :

$$2.8c - 0.372 = 5.1c - 2.21$$
$$5.1c - 2.8c = 2.21 - 0.372$$
$$2.3c = 1.838$$
$$c = \frac{1.838}{2.3}$$
$$c = 0.7991$$

Substitute the value of c in equation (i)

$$\ln k = 2.8c - 0.372$$
$$\ln k = 2.8 \times (0.7991) - 0.372$$
$$\ln k = 1.865$$
$$k = e^{1.865}$$
$$k = 6.5 \ (2sf)$$

$$\boxed{c = 0.80\,(2sf) \quad k = 6.5\,(2sf)}$$

15.

(a) Express $3\cos 2x - \sqrt{3}\sin 2x$ in the form $R\cos(2x+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give the exact values of R and α. [3]

$$3\cos 2x - \sqrt{3}\sin 2x = R\cos(2x+\alpha)$$

$$3\cos 2x - \sqrt{3}\sin 2x = R[\cos 2x\cos\alpha - \sin 2x\sin\alpha]$$

$$3\cos 2x - \sqrt{3}\sin 2x = R\cos 2x\cos\alpha - R\sin 2x\cdot\sin\alpha$$

Comparing the Coefficient:

$\boxed{\text{Formula: } \cos(A+B) = \cos A\cos B - \sin A\sin B}$

$R\cos\alpha = 3$ — (i) $\qquad R\sin\alpha = \sqrt{3}$ — (ii)

Dividing equation (ii) by equation (i)

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{\sqrt{3}}{3}$$

$$\tan\alpha = \frac{\sqrt{3}}{3}$$

$$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$$

$$\alpha = \frac{\pi}{6}$$

$R\cos\alpha = 3 \qquad R\sin\alpha = \sqrt{3}$

$R^2\cos^2\alpha = 9$ — (iii) $\qquad R^2\sin^2\alpha = 3$ — (iv)

Adding (i) and (ii)

$$R^2\sin^2\alpha + R^2\cos^2\alpha = 9+3$$

$$R^2(\sin^2\alpha + \cos^2\alpha) = 12 \qquad \sin^2\alpha + \cos^2\alpha = 1$$

$$R^2 = 12$$

$$R = \sqrt{12} = 2\sqrt{3}$$

$$\boxed{R\cos(2x+\alpha) = 2\sqrt{3}\cos\left(2x+\frac{\pi}{6}\right)}$$

(b) Hence find the exact value of $\int_0^{\frac{1}{12}\pi} \frac{3}{\left(3\cos 2x - \sqrt{3}\sin 2x\right)^2}\,dx$, simplifying your answer. [5]

$$\int_0^{\frac{1}{12}\pi} \frac{3}{(3\cos 2x - \sqrt{3}\sin 2x)^2} \cdot dx$$

$$\int_0^{\frac{\pi}{12}} \frac{3}{\left[2\sqrt{3}\cos\left(2x+\frac{\pi}{6}\right)\right]^2} \cdot dx$$

$$\int_0^{\pi/12} \frac{3}{12\cos^2\left(2x+\frac{\pi}{6}\right)} \cdot dx$$

$$\frac{3}{12}\int_0^{\pi/12} \sec^2\left(2x+\frac{\pi}{6}\right) dx$$

$\boxed{\frac{1}{\cos\theta} = \sec\theta}$

$$\frac{1}{4}\left[\frac{1}{2}\tan\left(2x+\frac{\pi}{6}\right)\right]_0^{\pi/12}$$

$\boxed{\text{Formula: } \int \sec^2\theta \cdot d\theta = \tan\theta}$

$$\frac{1}{8}\left[\tan\left(2x+\frac{\pi}{6}\right)\right]_0^{\pi/12}$$

$$\frac{1}{8}\left[\tan\left(2\times\frac{\pi}{12}+\frac{\pi}{6}\right) - \tan\left(0+\frac{\pi}{6}\right)\right]$$

$$\frac{1}{8}\left[\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{6}\right)\right]$$

$$\frac{1}{8}\left[\sqrt{3} - \frac{\sqrt{3}}{3}\right]$$

$$\boxed{\frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{24} = \frac{\sqrt{3}}{12}}$$

16.

Use the substitution $2x = \tan\theta$ to find the exact value of

$$\int_0^{\frac{1}{2}} \frac{12}{\left(1+4x^2\right)^2}\,dx .$$

Give your answer in the form $a + b\pi$, where a and b are rational numbers. [9]

$2x = \tan\theta$

$x = \frac{1}{2}\tan\theta$

$\frac{dx}{d\theta} = \frac{1}{2}\sec^2\theta$

$d\theta = \frac{1}{2}\sec^2\theta \cdot d\theta$

Change of Limits

$2x = \tan\theta$

$x=0,\quad 0 = \tan\theta \Rightarrow \theta = \tan^{-1}(0) = 0$

$x=\frac{1}{2},\quad 2\times\frac{1}{2} = \tan\theta \Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$

$$\int_0^{\frac{1}{2}} \frac{12}{(1+4x^2)^2}\,dx$$

$$\int_0^{\frac{1}{2}} \frac{12}{[1+(2x)^2]^2}\,dx$$

Given $2x = \tan\theta$

$$\int_0^{\pi/4} \frac{12}{(1+\tan^2\theta)^2}\,dx$$

$$\int_0^{\pi/4} \frac{12}{(\sec^2\theta)^2} \times \frac{1}{2}\sec^2\theta\,d\theta$$

$1+\tan^2\theta = \sec^2\theta$

$$\frac{12}{2}\int_0^{\pi/4} \frac{\cancel{\sec^2\theta}}{\sec^2\theta \times \cancel{\sec^2\theta}}\,d\theta$$

$$6\int_0^{\pi/4} \frac{1}{\sec^2\theta}\cdot d\theta$$

$$\frac{1}{\cos\theta} = \sec\theta$$

$$6\int_0^{\pi/4} \cos^2\theta \cdot d\theta$$

$$\boxed{\begin{aligned} \cos 2\theta &= 2\cos^2\theta - 1 \\ 2\cos^2\theta &= 1+\cos 2\theta \\ \cos^2\theta &= \frac{1+\cos 2\theta}{2} \end{aligned}}$$

$$6\int_0^{\pi/4} \frac{1+\cos 2\theta}{2} \cdot d\theta$$

$$3\int_0^{\pi/4} (1+\cos 2\theta)\, d\theta$$

$$3\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$3\left[\frac{\pi}{4} + \frac{1}{2}\times\sin\left(2\times\frac{\pi}{4}\right) - \left(0 - \frac{1}{2}\sin 0\right)\right]$$

$$3\left[\frac{\pi}{4} + \frac{1}{2}\right]$$

$$\boxed{\frac{3\pi}{4} + \frac{3}{2}}$$

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