1.

A curve has equation $y = x^2 + 2cx + 4$ and a straight line has equation $y = 4x + c$, where c is a constant.

Find the set of values of c for which the curve and line intersect at two distinct points. [5]

$y = x^2 + 2cx + 4$ $\qquad$ $y = 4x + c$

For Intersection : (Solve as Simultaneous Equations)

$$x^2 + 2cx + 4 = 4x + c$$

$$x^2 + 2cx - 4x + 4 - c = 0$$

$$\underbrace{x^2}_{a} + \underbrace{(2c-4)}_{b}x + \underbrace{(4-c)}_{c} = 0$$

For Two distinct point of intersection:

$$b^2 - 4ac > 0$$

$$(2c-4)^2 - 4 \times 1 (4-c) > 0$$

$$4c^2 - 16c + 16 - 4(4-c) > 0$$

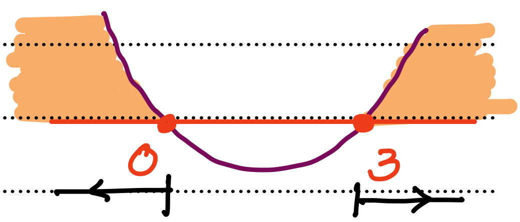
$$4c^2 - 16c + 16 - 16 + 4c > 0$$

$$4c^2 - 12c > 0$$

$$4c(c-3) > 0$$

For Critical values :

$4c = 0$	$c - 3 = 0$
$c = 0$	$c = 3$



$$\boxed{c > 3 \quad \text{or} \quad c < 0}$$

2.

Find the term independent of x in each of the following expansions.

(a) $\left(3x + \frac{2}{x^2}\right)^6$ [3]

$(a+b)^n = a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + {}^nC_3 a^{n-3} b^3 + \dots$

$\left(3x + \frac{2}{x^2}\right)^6 = (3x)^6 + {}^6C_1 (3x)^5 \left(\frac{2}{x^2}\right)^1 + {}^6C_2 (3x)^4 \left(\frac{2}{x^2}\right)^2 + {}^6C_3 (3x)^3 \left(\frac{2}{x^2}\right)^3 + {}^6C_4 (3x)^2 \left(\frac{2}{x^2}\right)^4 + \dots$

$15 \times 81 x^4 \times \frac{4}{x^4}$

$= 15 \times 81 \times 4$ (independent of x)

$= 4860$

(b) $\left(3x + \frac{2}{x^2}\right)^6 (1 - x^3)$ [3]

$\left(3x + \frac{2}{x^2}\right)^6 = (3x)^6 + {}^6C_1 (3x)^5 \left(\frac{2}{x^2}\right)^1 + {}^6C_2 (3x)^4 \left(\frac{2}{x^2}\right)^2 + {}^6C_3 (3x)^3 \left(\frac{2}{x^2}\right)^3 + {}^6C_4 (3x)^2 \left(\frac{2}{x^2}\right)^4 + \dots$

4860

$20 \times 27 x^3 \times \frac{8}{x^6}$

$4320 \times \frac{1}{x^3}$

$\left(3x + \frac{2}{x^2}\right)^6 (1 - x^3)$

$\left(4860 + 4320 \times \frac{1}{x^3}\right)(1 - x^3)$

(Ignore other terms)

$4860 - 4320 = 540$

3.

(a) Express $2x^2 - 8x + 14$ in the form $2[(x-a)^2 + b]$. [2]

$2x^2 - 8x + 14$

$2[x^2 - 4x + 7]$

$2[(x-2)^2 - 2^2 + 7]$

$2[(x-2)^2 - 4 + 7]$

$2[(x-2)^2 + 3]$

The functions f and g are defined by

$$f(x) = x^2 \quad \text{for } x \in \mathbb{R},$$
$$g(x) = 2x^2 - 8x + 14 \quad \text{for } x \in \mathbb{R}.$$

(b) Describe fully a sequence of transformations that maps the graph of $y = f(x)$ onto the graph of $y = g(x)$, making clear the order in which the transformations are applied. [4]

$f(x) = x^2$ $g(x) = 2x^2 - 8x + 14$

$g(x) = 2[(x-2)^2 + 3]$ from part-a

$f(x) = x^2$

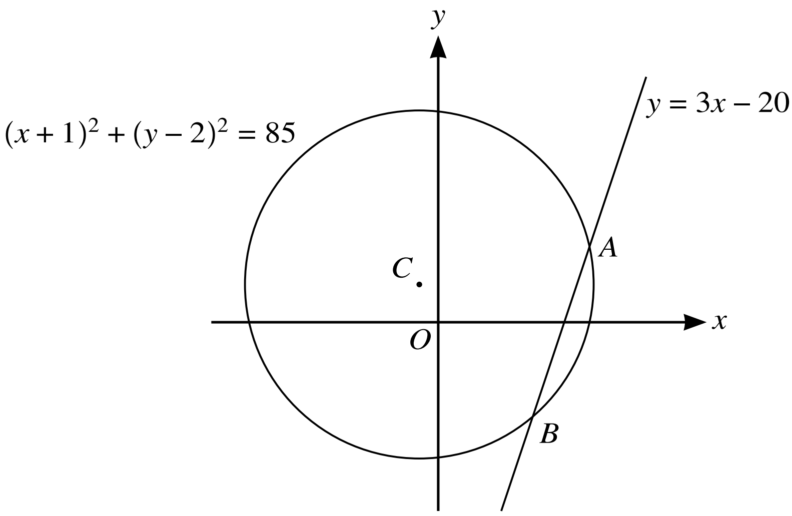
1. $f(x) = (x-2)^2 + 3$ Translation by $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$
2. $f(x) = 2[(x-2)^2 + 3]$ Stretch parallel to y-axis with scale factor 2.

Alternative method: $f(x) = x^2$ $g(x) = 2[(x-2)^2 + 3]$

$f(x) = x^2$ $g(x) = 2(x-2)^2 + 6$

1. $f(x) = 2x^2$ Stretch parallel to y-axis scale factor 2
2. $f(x) = 2(x-2)^2 + 6$ Translation by $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$

4.



The circle with equation $(x + 1)^2 + (y - 2)^2 = 85$ and the straight line with equation $y = 3x - 20$ are shown in the diagram. The line intersects the circle at A and B, and the centre of the circle is at C.

(a) Find, by calculation, the coordinates of A and B. [4]

$(x+1)^2 + (y-2)^2 = 85$ $\quad y = 3x - 20$

$(x+1)^2 + (3x-20-2)^2 = 85$

$(x+1)^2 + (3x-22)^2 = 85$

$(x+1)(x+1) + (3x-22)(3x-22) = 85$

$x^2 + x + x + 1 + 9x^2 - 66x - 66x + 484 = 85$

$10x^2 - 130x + 485 = 85$

$10x^2 - 130x + 485 - 85 = 0$

$10x^2 - 130x + 400 = 0$ Divide by 10 in each term

$x^2 - 13x + 40 = 0$

$(x-8)(x-5) = 0$

$x - 8 = 0$	$x - 5 = 0$
$x = 8$	$x = 5$
$y = 3x - 20$	$y = 3x - 20$
$y = 3 \times 8 - 20$	$y = 3 \times 5 - 20$
$y = 4$	$y = -5$

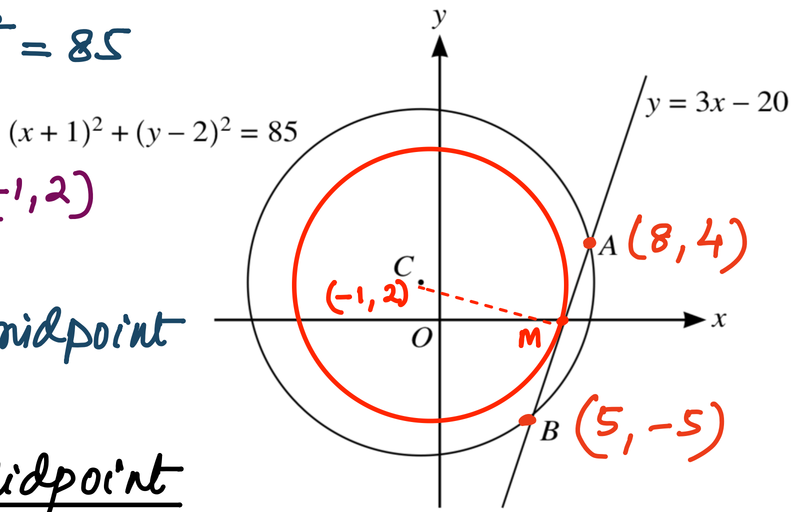
Coordinates of A and B

$\boxed{(8, 4) \text{ and } (5, -5)}$

(b) Find an equation of the circle which has its centre at C and for which the line with equation $y = 3x - 20$ is a tangent to the circle. [4]

Equation of Circle:

$(x+1)^2 + (y-2)^2 = 85$



Centre of Circle: $(-1, 2)$

Point M is the midpoint of A and B

Coordinates of Midpoint

$$M\left[\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right]$$

$$M\left[\frac{8+5}{2}, \frac{4+(-5)}{2}\right]$$

$$M\left[\frac{13}{2}, \frac{-1}{2}\right]$$

Point M is lying on new Circle and new Circle has same Centre of older one.

Equation of Circle:

$$(x+1)^2 + (y-2)^2 = r^2$$

$$\left(\frac{13}{2}+1\right)^2 + \left(\frac{-1}{2}-2\right)^2 = r^2$$

$$\left(\frac{15}{2}\right)^2 + \left(\frac{-5}{2}\right)^2 = r^2$$

$$\frac{225}{4} + \frac{25}{4} = r^2$$

$$r^2 = \frac{250}{4} = \frac{125}{2}$$

$$\boxed{(x+1)^2 + (y-2)^2 = \frac{125}{2}}$$

5.

(a) Show that $\dfrac{\sin\theta + 2\cos\theta}{\cos\theta - 2\sin\theta} - \dfrac{\sin\theta - 2\cos\theta}{\cos\theta + 2\sin\theta} \equiv \dfrac{4}{5\cos^2\theta - 4}$. [4]

$$\frac{\sin\theta + 2\cos\theta}{\cos\theta - 2\sin\theta} - \frac{\sin\theta - 2\cos\theta}{\cos\theta + 2\sin\theta}$$

$$\frac{(\sin\theta + 2\cos\theta)(\cos\theta + 2\sin\theta) - (\sin\theta - 2\cos\theta)(\cos\theta - 2\sin\theta)}{(\cos\theta - 2\sin\theta)(\cos\theta + 2\sin\theta)}$$

$$\frac{\cancel{\sin\theta\cos\theta} + 2\sin^2\theta + 2\cos^2\theta + \cancel{4\sin\theta\cos\theta} - \cancel{\sin\theta\cos\theta} + 2\sin^2\theta + 2\cos^2\theta - \cancel{4\sin\theta\cos\theta}}{\cos^2\theta + \cancel{2\sin\theta\cos\theta} - \cancel{2\sin\theta\cdot\cos\theta} - 4\sin^2\theta}$$

$$\frac{4\sin^2\theta + 4\cos^2\theta}{\cos^2\theta - 4\sin^2\theta}$$

$$\frac{4(\sin^2\theta + \cos^2\theta)}{\cos^2\theta - 4(1 - \cos^2\theta)}$$

$$\frac{4 \times 1}{\cos^2\theta - 4 + 4\cos^2\theta}$$ $\sin^2\theta + \cos^2\theta = 1$

$$\frac{4}{5\cos^2\theta - 4}$$

(b) Hence solve the equation $\dfrac{\sin\theta + 2\cos\theta}{\cos\theta - 2\sin\theta} - \dfrac{\sin\theta - 2\cos\theta}{\cos\theta + 2\sin\theta} = 5$ for $0° < \theta < 180°$. [3]

$$\frac{4}{5\cos^2\theta - 4} = 5$$

$$4 = 5(5\cos^2\theta - 4)$$

$$4 = 25\cos^2\theta - 20$$

$$25\cos^2\theta = 24$$

$$\cos^2\theta = \frac{24}{25}$$

$$\cos\theta = \pm\sqrt{\frac{24}{25}}$$

$\cos\theta = \sqrt{\frac{24}{25}}$ $\qquad$ $\cos\theta = -\sqrt{\frac{24}{25}}$

$\theta = \cos^{-1}\left(\sqrt{\frac{24}{25}}\right)$ $\qquad$ $\theta = \cos^{-1}\left(-\sqrt{\frac{24}{25}}\right)$

$\theta = 11.5°$ $\qquad$ $\theta = 168.46°$

$$\boxed{\theta = 11.5°,\ 168.5°}$$

6.

Functions f, g and h are defined as follows:

$$f : x \mapsto x - 4x^{\frac{1}{2}} + 1 \quad \text{for } x \geqslant 0,$$
$$g : x \mapsto mx^2 + n \quad \text{for } x \geqslant -2, \text{ where } m \text{ and } n \text{ are constants,}$$
$$h : x \mapsto x^{\frac{1}{2}} - 2 \quad \text{for } x \geqslant 0.$$

(a) Solve the equation $f(x) = 0$, giving your solutions in the form $x = a + b\sqrt{c}$, where a, b and c are integers. [4]

$f(x) = 0$

$x - 4x^{\frac{1}{2}} + 1 = 0$

Let $y = x^{1/2}$

$y^2 = x$

$y^2 - 4y + 1 = 0$

$y = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2 \times 1}$

$y = \dfrac{4 \pm \sqrt{12}}{2}$

$y = \dfrac{4 \pm 2\sqrt{3}}{2} = \dfrac{\cancel{2}(2 \pm \sqrt{3})}{\cancel{2}}$

$y = 2 \pm \sqrt{3}$

$x = y^2$

$x = (2+\sqrt{3})^2$

$x = (2+\sqrt{3})(2+\sqrt{3})$

$x = 4 + 2\sqrt{3} + 2\sqrt{3} + 3$

$\boxed{x = 7 + 4\sqrt{3}}$

$x = (2-\sqrt{3})^2$

$x = (2-\sqrt{3})(2-\sqrt{3})$

$x = 4 - 2\sqrt{3} - 2\sqrt{3} + 3$

$\boxed{x = 7 - 4\sqrt{3}}$

(b) Given that $f(x) \equiv gh(x)$, find the values of m and n. [4]

$g(x) = mx^2 + n$ $\quad h(x) = x^{\frac{1}{2}} - 2$

$gh(x) = m[h(x)]^2 + n$

$gh(x) = m(x^{\frac{1}{2}} - 2)^2 + n$

$f(x) \equiv gh(x)$

$x - 4x^{\frac{1}{2}} + 1 \equiv m(x^{1/2} - 2)^2 + n$

$x - 4x^{1/2} + 1 \equiv m(x^{1/2} - 2)(x^{1/2} - 2) + n$

$x - 4x^{1/2} + 1 \equiv m(x - 2x^{1/2} - 2x^{1/2} + 4) + n$

$x - 4x^{1/2} + 1 \equiv m(x - 4x^{1/2} + 4) + n$

$x - 4x^{1/2} + 1 \equiv mx - 4mx^{1/2} + 4m + n$

Comparing the coefficients:

$m = 1$

$4m + n = 1$

$4(1) + n = 1$

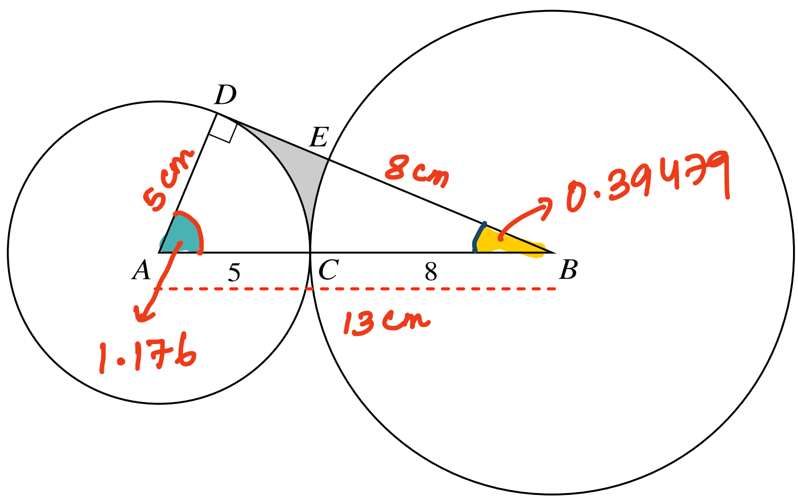
$4 + n = 1$

$n = 1 - 4$

$n = -3$

$\boxed{m = 1 \qquad n = -3}$

7.



The diagram shows a circle with centre A of radius 5 cm and a circle with centre B of radius 8 cm. The circles touch at the point C so that ACB is a straight line. The tangent at the point D on the smaller circle intersects the larger circle at E and passes through B.

(a) Find the perimeter of the shaded region. [5]

In right angled triangle ADB :

$BD = \sqrt{13^2 - 5^2}$ (Pythogoras theorem)

$BD = 12$ cm

$DE = 12 - 8$

$DE = 4$ cm

To find angle ABD

$\tan(D\hat{A}B) = \dfrac{12}{5}$

$D\hat{A}B = \tan^{-1}\left(\dfrac{12}{5}\right)$

$D\hat{A}B = 1.176$

$A\hat{B}D = \dfrac{\pi}{2} - 1.176$

$A\hat{B}D = 0.39479$

Arc length = $r\theta$

Perimeter of shaded Region :

DE + Arc length DC + Arc length EC

$4 + (5 \times 1.176) + (8 \times 0.39479)$

13.038

$= 13$ cm

(b) Find the area of the shaded region. [3]

Area of shaded Region :

$\boxed{\text{Area of Sector} = \frac{1}{2}r^2\theta}$

Area of triangle ABD − [Area of Sector ACD + Area of Sector BCE]

$\frac{1}{2} \times 5 \times 12 - \left[\frac{1}{2} \times 5^2 \times 1.176 + \frac{1}{2} \times 8^2 \times 0.39479\right]$

$30 - 27.33328$

2.666

$= 2.7$ (3sf)

8.

The coefficient of x^4 in the expansion of $(3+x)^5$ is equal to the coefficient of x^2 in the expansion of $\left(2x+\frac{a}{x}\right)^6$.

Find the value of the positive constant a. [4]

$(a+b)^n = a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \dots$

$(3+x)^5 = 3^5 + {}^5C_1 3^4 x + {}^5C_2 3^3 x^2 + {}^5C_3 3^2 x^3 + {}^5C_4 3^1 x^4$

$5 \times 3 \times x^4$

$15x^4$

Coefficient of $x^4 = 15$

$\left(2x+\frac{a}{x}\right)^6 = (2x)^6 + {}^6C_1 (2x)^5 \left(\frac{a}{x}\right) + {}^6C_2 (2x)^4 \left(\frac{a}{x}\right)^2 + {}^6C_3 (2x)^3 \left(\frac{a}{x}\right)^3 + \dots$

$15 \times 16x^4 \times \frac{a^2}{x^2}$

$240a^2x^2$

Coefficient of $x^2 = 240a^2$

As per Question: $240a^2 = 15$

$a^2 = \frac{15}{240}$

$a = \sqrt{\frac{15}{240}}$

$\boxed{a = \frac{1}{4}}$

9.

The equation of a curve is $y = 4x^2 - kx + \frac{1}{2}k^2$ and the equation of a line is $y = x - a$, where k and a are constants.

(a) Given that the curve and the line intersect at the points with x-coordinates 0 and $\frac{3}{4}$, find the values of k and a. [4]

For intersection, Solve as Simultaneous Equations

$$y = 4x^2 - kx + \frac{1}{2}k^2$$
$$y = x - a$$

$$4x^2 - kx + \frac{1}{2}k^2 = x - a$$
$$4x^2 - kx + \frac{1}{2}k^2 - x + a = 0$$
$$4x^2 - kx - x + \frac{1}{2}k^2 + a = 0$$
$$4x^2 - (k+1)x + \frac{1}{2}k^2 + a = 0$$

For $x = 0$,

$$4(0) - (k+1)(0) + \frac{1}{2}k^2 + a = 0$$
$$0 - 0 + \frac{1}{2}k^2 + a = 0$$
$$a = -\frac{1}{2}k^2$$

For $x = \frac{3}{4}$,

$$4\left(\frac{3}{4}\right)^2 - (k+1)\left(\frac{3}{4}\right) + \frac{1}{2}k^2 + a = 0$$
$$\frac{9}{4} - \frac{3k}{4} - \frac{3}{4} + \frac{1}{2}k^2 + a = 0$$
$$\frac{6}{4} - \frac{3k}{4} + \frac{1}{2}k^2 - \frac{1}{2}k^2 = 0$$
$$\frac{3k}{4} = \frac{6}{4}$$
$$\boxed{k = 2}$$

$$a = -\frac{1}{2}k^2$$
$$a = -\frac{1}{2} \times 2^2$$
$$\boxed{a = -2}$$

(b) Given instead that $a = -\frac{7}{2}$, find the values of k for which the line is a tangent to the curve. [5]

$4x^2 - (k+1)x + \frac{1}{2}k^2 + a = 0$

$a = \frac{-7}{2}$,

$\underbrace{4}_{a}x^2 \underbrace{- (k+1)}_{b}x + \underbrace{\frac{1}{2}k^2 - \frac{7}{2}}_{c} = 0$

for tangency: $b^2 - 4ac = 0$

$[-(k+1)]^2 - 4 \times 4 \times \left(\frac{1}{2}k^2 - \frac{7}{2}\right) = 0$

$(k+1)^2 - 16\left(\frac{1}{2}k^2 - \frac{7}{2}\right) = 0$

$k^2 + 2k + 1 - 8k^2 + 56 = 0$

$-7k^2 + 2k + 57 = 0$

$7k^2 - 2k - 57 = 0$

$k = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(7)(-57)}}{2 \times 7}$

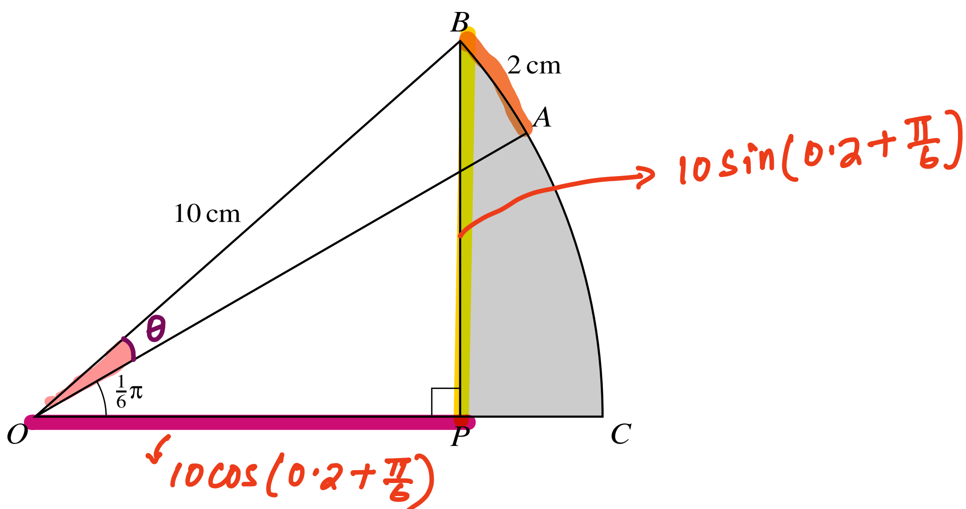
$k = \frac{2 \pm \sqrt{1600}}{14}$

$k = \frac{2 \pm 40}{14}$

$k = \frac{2+40}{14}$ $\qquad$ $k = \frac{2-40}{14}$

$\boxed{k = 3}$ $\qquad$ $\boxed{k = \frac{-19}{7}}$

10.



The diagram shows a sector $OBAC$ of a circle with centre O and radius 10 cm. The point P lies on OC and BP is perpendicular to OC. Angle $AOC = \frac{1}{6}\pi$ and the length of the arc AB is 2 cm.

(a) Find the angle BOC. [2]

Arc length $AB = 2$ cm

$r\theta = 2$

$10 \times \theta = 2$

$\theta = \frac{2}{10}$

$\theta = 0{\cdot}2$

Angle $BOC = \left(0{\cdot}2 + \frac{\pi}{6}\right)$ radians

(b) Hence find the area of the shaded region BPC giving your answer correct to 3 significant figures. [4]

In triangle OBP

$$\sin\left(0\cdot2+\frac{\pi}{6}\right) = \frac{PB}{10}$$ (using SOHCAHTOA)

$$PB = 10\sin\left(0\cdot2+\frac{\pi}{6}\right)$$

$$\cos\left(0\cdot2+\frac{\pi}{6}\right) = \frac{OP}{10}$$

$$OP = 10\cos\left(0\cdot2+\frac{\pi}{6}\right)$$

Shaded Area = Area of Sector OBC − Area of triangle OBP

$$= \frac{1}{2}\times10^2\left(0\cdot2+\frac{\pi}{6}\right) - \frac{1}{2}\times10\cos\left(0\cdot2+\frac{\pi}{6}\right)\times10\sin\left(0\cdot2+\frac{\pi}{6}\right)$$

$$= 11\cdot37065$$

$$= 11\cdot4 \text{ cm}^2$$

11.

The equation of a circle is $x^2 + y^2 + ax + by - 12 = 0$. The points $A(1, 1)$ and $B(2, -6)$ lie on the circle.

(a) Find the values of a and b and hence find the coordinates of the centre of the circle. [4]

$x^2 + y^2 + ax + by - 12 = 0$

$A(1,1)$ $\quad 1^2 + 1^2 + a\times 1 + b\times 1 - 12 = 0$

$2 + a + b = 12$

$a + b = 10$ —— (i)

$B(2,-6)$ $\quad 2^2 + (-6)^2 + a(2) + b(-6) - 12 = 0$

$4 + 36 + 2a - 6b - 12 = 0$

$2a - 6b = 12 - 40$

$2a - 6b = -28$

$a - 3b = -14$ —— (ii)

Solving Equations (i) and (ii)

$a + b = 10$

$-\; a \mp 3b = \mp 14$

$4b = 24$

$b = 6$

$a + b = 10$

$a + 6 = 10$

$a = 4$

For $a = 4$, $b = 6$

$x^2 + y^2 + 4x + 6y - 12 = 0$

$x^2 + 4x + y^2 + 6y = 12$

Using Completing the square

$(x+2)^2 - 2^2 + (y+3)^2 - 3^2 = 12$

$(x+2)^2 - 4 + (y+3)^2 - 9 = 12$

$(x+2)^2 + (y+3)^2 = 12 + 4 + 9$

$(x+2)^2 + (y+3)^2 = 5^2$

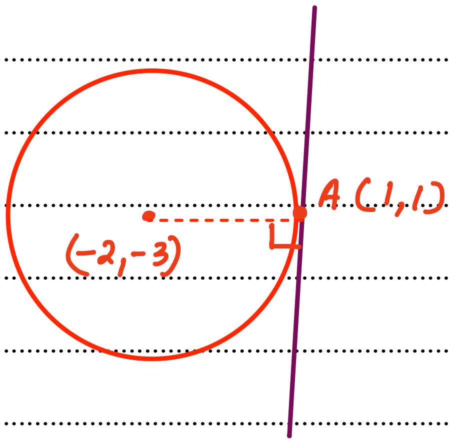
Centre of Circle : $(-2, -3)$

(b) Find the equation of the tangent to the circle at the point A, giving your answer in the form $px + qy = k$, where p, q and k are integers. [4]

Gradient of radius $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$m_{radius} = \dfrac{1-(-3)}{1-(-2)}$

$m_{radius} = \dfrac{4}{3}$



gradient of tangent $(m_{tangent}) = \dfrac{-3}{4}$

(tangent and radius are perpendicular : $m_1 \times m_2 = -1$)

Equation of tangent :

$y - y_1 = m(x - x_1)$

at A(1,1) $\quad y - 1 = -\dfrac{3}{4}(x-1)$

$4(y-1) = -3(x-1)$

$4y - 4 = -3x + 3$

$3x + 4y = 3 + 4$

$3x + 4y = 7$

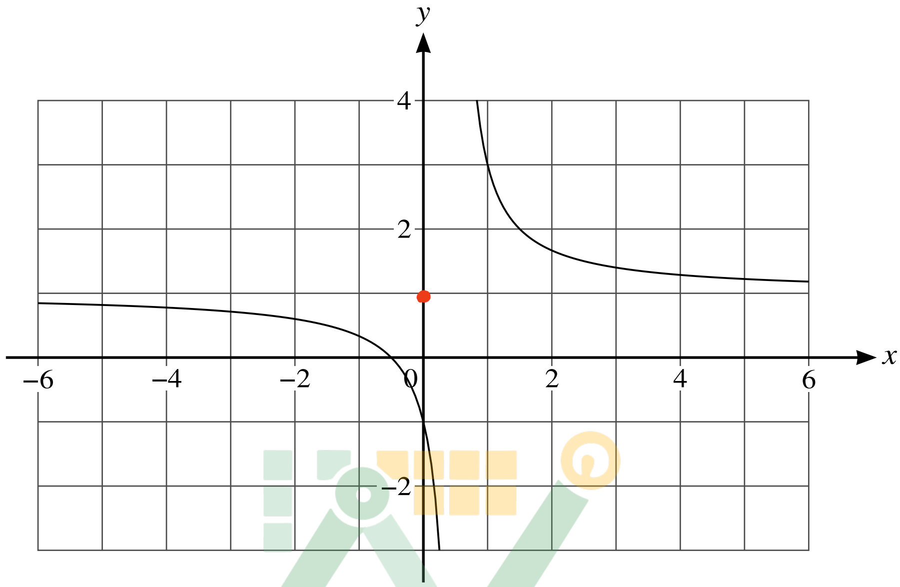
12.

Functions f and g are defined as follows:

$$f(x) = \frac{2x+1}{2x-1} \quad \text{for } x \neq \tfrac{1}{2},$$

$$g(x) = x^2 + 4 \quad \text{for } x \in \mathbb{R}.$$

(a)



The diagram shows part of the graph of $y = f(x)$.

State the domain of f^{-1}. [1]

check in y axis

Domain of $f^{-1}(x)$ = Range of $f(x)$, The graph $f(x)$ is approaching to 1, never equals to 1. All values of $f(x)$ do not pass through $y = 1$. Domain of $f^{-1}(x)$: $x \neq 1$

(b) Find an expression for $f^{-1}(x)$. [3]

$y = \frac{2x+1}{2x-1}$

$y(2x-1) = 2x+1$

$2xy - y = 2x+1$

$2xy - 2x = 1+y$

$x(2y-2) = 1+y$

$x = \frac{1+y}{2y-2}$

$\boxed{f^{-1}(x) = \frac{1+x}{2x-2}}$ $\quad x \neq 1$

(c) Find $gf^{-1}(3)$. [2]

$f^{-1}(3) = \frac{1+3}{2\times3-2} = 1$

$g(x) = x^2+4$

$gf^{-1}(3) = g(1) = 1^2+4 = 5$

(d) Explain why $g^{-1}(x)$ cannot be found. [1]

$g^{-1}(x)$ Can not be found because $g(x)$ is not a one to one function.

Example: $g(1) = 1^2+4 = 5$ $\quad g(-1) = (-1)^2+4 = 5$

(e) Show that $1 + \dfrac{2}{2x-1}$ can be expressed as $\dfrac{2x+1}{2x-1}$. Hence find the area of the triangle enclosed by the tangent to the curve $y = f(x)$ at the point where $x = 1$ and the x- and y-axes. [6]

$$1 + \frac{2}{2x-1} = \frac{2x-1+2}{2x-1}$$

$$= \frac{2x+1}{2x-1}$$

$$f(x) = 1 + \frac{2}{2x-1}$$

$$f(x) = 1 + 2(2x-1)^{-1}$$

$$f'(x) = 0 + 2(-1)(2x-1)^{-2}(2)$$

$$= -4(2x-1)^{-2}$$

$$= \frac{-4}{(2x-1)^2}$$

at $x=1$, $f'(x) = \dfrac{-4}{(2\times1-1)^2} = 4$

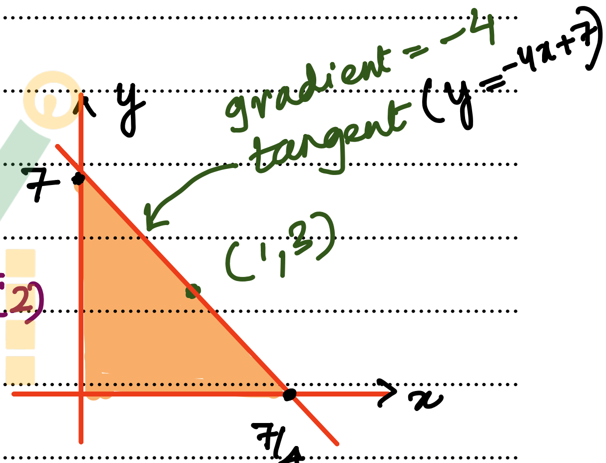
Gradient of tangent $= -4$

find the Coordinate:

$$y = 1 + \frac{2}{2x-1}$$

$x=1$, $\quad y = 1 + \dfrac{2}{2\times1-1}$

$y = 3 \qquad (1, 3)$



Equation of tangent:

$y - y_1 = m(x - x_1)$

$y - 3 = -4(x-1)$

$y - 3 = -4x + 4$

$y = -4x + 7$

On x axis ($y=0$)

$0 = -4x + 7 \Rightarrow x = 7/4$

On y axis ($x=0$)

$y = -4(0) + 7 \Rightarrow y = 7$

$$\boxed{\text{Area of Triangle: } \tfrac{1}{2} \times \tfrac{7}{4} \times 7 = \tfrac{49}{8}}$$

13.

The function f is given by $f(x) = 4\cos^4 x + \cos^2 x - k$ for $0 \leqslant x \leqslant 2\pi$, where k is a constant.

(a) Given that $k = 3$, find the exact solutions of the equation $f(x) = 0$. [5]

$f(x) = 0$

$4\cos^4 x + \cos^2 x - k = 0$

$4\cos^4 x + \cos^2 x - 3 = 0$ Given $k = 3$

Let $y = \cos^2 x$

$y^2 = \cos^4 x$

$4y^2 + y - 3 = 0$

$(4y - 3)(y + 1) = 0$

$4y - 3 = 0$ $\qquad$ $y + 1 = 0$

$y = \frac{3}{4}$ $\qquad$ $y = -1$

$\cos^2 x = -1$

$\cos^2 x = \frac{3}{4}$ $\qquad$ $\cos x = \sqrt{-1}$

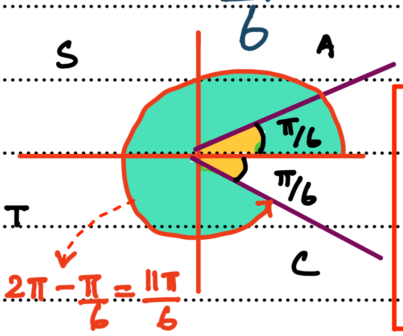
(NO SOLUTIONS)

$\cos x = \pm\sqrt{\frac{3}{4}}$

$\cos x = \sqrt{\frac{3}{4}}$ $\qquad$ $\cos x = -\sqrt{\frac{3}{4}}$

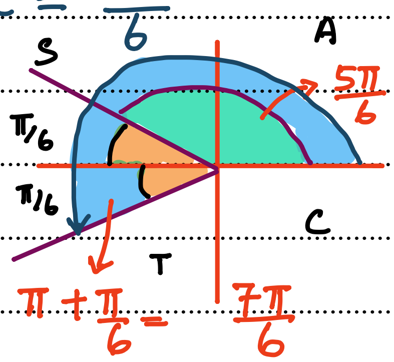
$x = \cos^{-1}\left(\sqrt{\frac{3}{4}}\right)$ $\qquad$ $x = \cos^{-1}\left(\sqrt{\frac{3}{4}}\right)$

$x = \frac{\pi}{6}$ $\qquad$ $x = \frac{5\pi}{6}$



$$\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}}$$

Four solutions



(b) Use the quadratic formula to show that, when $k > 5$, the equation $f(x) = 0$ has no solutions. [5]

$$4\cos^4 x + \cos^2 x - k = 0$$

Let $y = \cos^2 x$

$y^2 = \cos^4 x$

$$4y^2 + y - k = 0$$

$a = 4$, $b = 1$, $c = -k$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{-1 \pm \sqrt{1^2 - 4 \times 4 \times (-k)}}{2 \times 4}$$

$$y = \frac{-1 \pm \sqrt{1 + 16k}}{8}$$

$$\cos^2 x = \frac{-1 \pm \sqrt{1+16k}}{8}$$

$-1 \leq \cos x \leq 1$

$-1 \leq \cos^2 x \leq 1$

$$\frac{-1 \pm \sqrt{1+16k}}{8} \leq 1$$

$$-1 \pm \sqrt{1+16k} \leq 8$$

$$\pm\sqrt{1+6k} \leq 9$$

$\pm\sqrt{1+6k} \leq 9$ (Squaring on both sides)

$1 + 6k \leq 9^2$

$6k \leq 81 - 1$

$6k \leq 80$

$k \leq 5$ (Have Solutions)

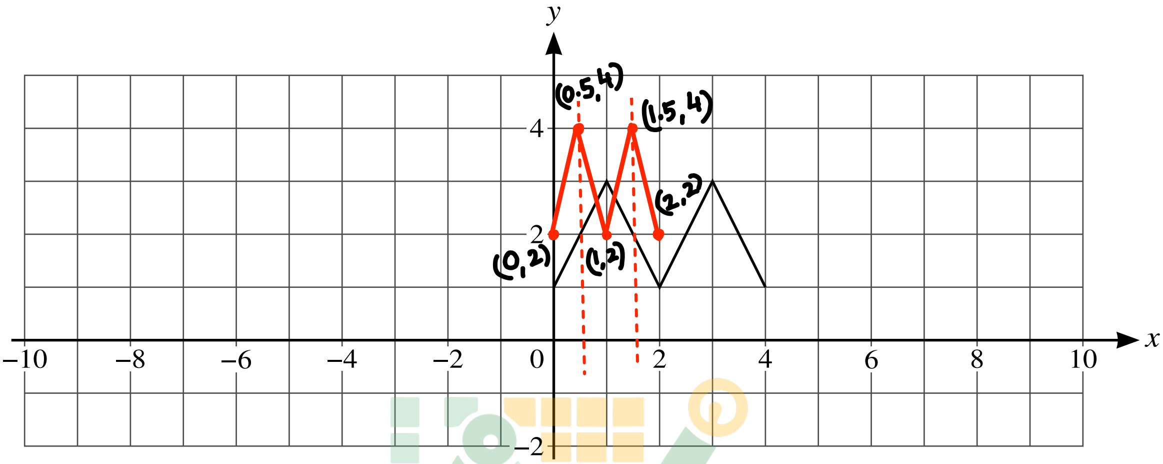
Therefore when $k > 5$, the equation has no solutions.

14.

The graph with equation $y = f(x)$ is transformed to the graph with equation $y = g(x)$ by a stretch in the x-direction with factor 0.5, followed by a translation of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(a) The diagram below shows the graph of $y = f(x)$.

On the diagram sketch the graph of $y = g(x)$. [3]



(b) Find an expression for $g(x)$ in terms of $f(x)$. [2]

$y = f(x)$ $\quad (x, y)$

1. A strech in x-direction with factor 0.5

$y = f\left(\frac{x}{0.5}\right)$

$y = f(2x)$ $\quad \left(\frac{x}{2}, y\right)$

2. Translation by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$y = f(2x) + 1$ $\quad \boxed{\left(\frac{x}{b}, y+1\right)}$

(x, y)	$(0, 1)$	$(1, 3)$	$(2, 1)$	$(3, 3)$	$(4, 1)$
$\left(\frac{x}{2}, y+1\right)$	$(0, 2)$	$(0.5, 4)$	$(1, 2)$	$(1.5, 4)$	$(2, 2)$

Plot this Coordinates on the graph.

15.

(a) Prove the identity $\dfrac{\sin\theta}{\sin\theta+\cos\theta}+\dfrac{\cos\theta}{\sin\theta-\cos\theta}\equiv\dfrac{\tan^2\theta+1}{\tan^2\theta-1}$. [3]

$$\frac{\sin\theta(\sin\theta-\cos\theta)+\cos\theta(\sin\theta+\cos\theta)}{(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}$$

$$\frac{\sin^2\theta-\cancel{\sin\theta\cdot\cos\theta}+\cancel{\sin\theta\cdot\cos\theta}+\cos^2\theta}{\sin^2\theta-\cancel{\sin\theta\cdot\cos\theta}+\cancel{\sin\theta\cdot\cos\theta}-\cos^2\theta}$$

$$\frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta-\cos^2\theta}$$

Divide $\cos^2\theta$ in both numerator and denominator

$$\frac{\dfrac{\sin^2\theta}{\cos^2\theta}+\dfrac{\cos^2\theta}{\cos^2\theta}}{\dfrac{\sin^2\theta}{\cos^2\theta}-\dfrac{\cos^2\theta}{\cos^2\theta}}$$

$$\frac{\tan^2\theta+1}{\tan^2\theta-1}$$

(b) Hence find the exact solutions of the equation $\frac{\sin\theta}{\sin\theta+\cos\theta}+\frac{\cos\theta}{\sin\theta-\cos\theta}=2$ for $0 \leqslant \theta \leqslant \pi$. [4]

$$\frac{\tan^2\theta+1}{\tan^2\theta-1}=2$$

$$\tan^2\theta+1 = 2(\tan^2\theta-1)$$

$$\tan^2\theta+1 = 2\tan^2\theta-2$$

$$2\tan^2\theta-\tan^2\theta = 1+2$$

$$\tan^2\theta = 3$$

$$\tan\theta = \pm\sqrt{3}$$

$\tan\theta = \sqrt{3}$

$\theta = \tan^{-1}\sqrt{3}$

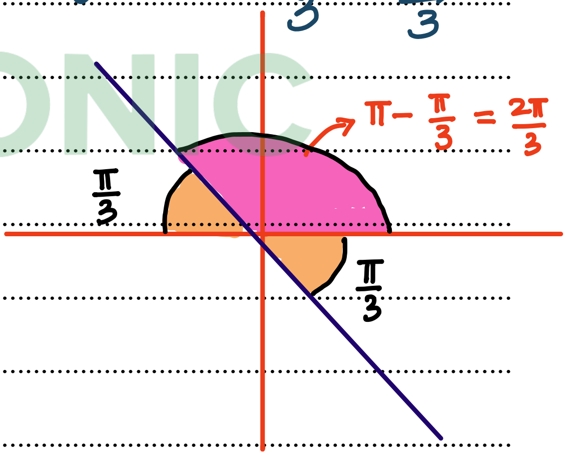
$\theta = \frac{\pi}{3}$

$\tan\theta = -\sqrt{3}$

$\theta = \tan^{-1}(-\sqrt{3})$

$\theta = \frac{-\pi}{3}$ (in fourth Quadrant)

$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$



$$\theta = \frac{\pi}{3}, \frac{2\pi}{3}$$

16.

Functions f and g are defined by

$$f(x) = x + \frac{1}{x} \quad \text{for } x > 0,$$
$$g(x) = ax + 1 \quad \text{for } x \in \mathbb{R},$$

where a is a constant.

(a) Find an expression for $gf(x)$. [1]

$g(x) = ax + 1$

$g\,f(x) = a\,f(x) + 1$

$g\,f(x) = a\left[x + \frac{1}{x}\right] + 1$

$= ax + \frac{a}{x} + 1$

(b) Given that $gf(2) = 11$, find the value of a. [2]

$gf(x) = ax + \frac{a}{x} + 1$

$gf(2) = 2a + \frac{a}{2} + 1$

$11 = 2a + \frac{a}{2} + 1$

$11 - 1 = 2a + \frac{a}{2}$

Multiply by 2 with each term

$10 = 2a + \frac{a}{2}$

$20 = 4a + a$

$5a = 20$

$a = 4$

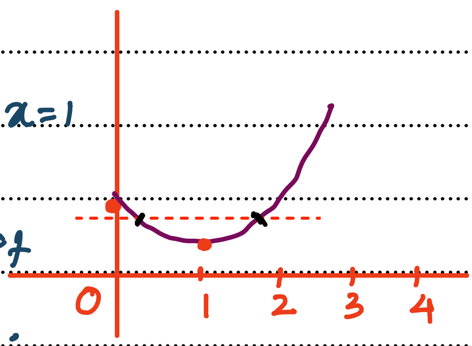
(c) Given that the graph of $y = f(x)$ has a minimum point when $x = 1$, explain whether or not f has an inverse. [1]

The graph of the function f(x) is shown.

The domain of f(x), $x > 0$

It has a minimum point when $x = 1$

which means when a horizontal line (Parallel to x-axis is drawn it intersect the graph at two points.



This is a many to one function. So f(x) has no inverse.

It is given instead that $a = 5$.

(d) Find and simplify an expression for $g^{-1}f(x)$. [3]

$g(x) = 5x+1$

$y = 5x+1$

$y-1 = 5x$

$x = \frac{y-1}{5}$

$g^{-1}(x) = \frac{x-1}{5}$

$g^{-1}f(x) = \frac{f(x)-1}{5}$

$= \frac{x+\frac{1}{x}-1}{5}$

$= \frac{\frac{x^2+1-x}{x}}{5}$

$\frac{x^2+1-x}{x} \div 5$

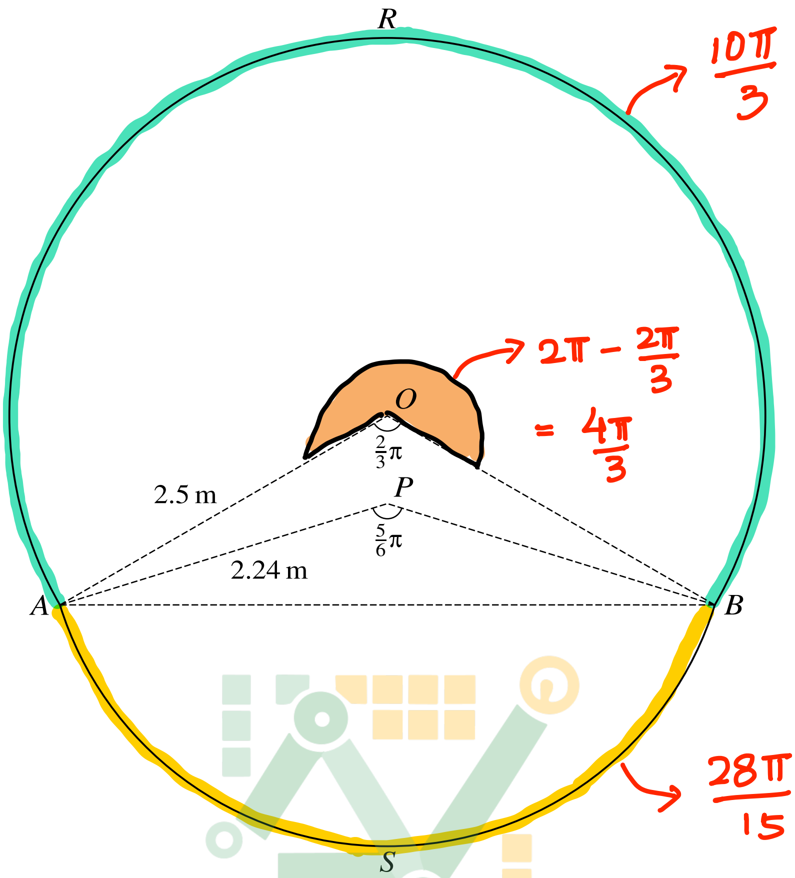
$\frac{x^2+1-x}{x} \times \frac{1}{5}$

$= \frac{x^2+1-x}{5x} = \frac{x^2-x+1}{5x}$

(e) Explain why the composite function fg cannot be formed. [1]

Range of $g(x)$ is not a subset of Domain of $f(x)$.
So the composite function $fg(x)$ can't be formed.

Range of $g(x)$ = Real numbers
Domain of $f(x)$: $x > 0$
which means negative numbers is not part of range of $g(x)$.



The diagram shows a cross-section $RASB$ of the body of an aircraft. The cross-section consists of a sector $OARB$ of a circle of radius 2.5 m, with centre O, a sector $PASB$ of another circle of radius 2.24 m with centre P and a quadrilateral $OAPB$. Angle $AOB = \frac{2}{3}\pi$ and angle $APB = \frac{5}{6}\pi$.

(a) Find the perimeter of the cross-section $RASB$, giving your answer correct to 2 decimal places. [3]

Arc $\widehat{ASB} = 2.24 \times \frac{5\pi}{6}$

$= \frac{28\pi}{15}$

Formula:
Arc length $= r\theta$

Arc $\widehat{ARB} = 2.5 \times \frac{4\pi}{3}$

$= \frac{10\pi}{3}$

Perimeter of Cross Section RASB : $\widehat{ASB} + \widehat{ARB}$

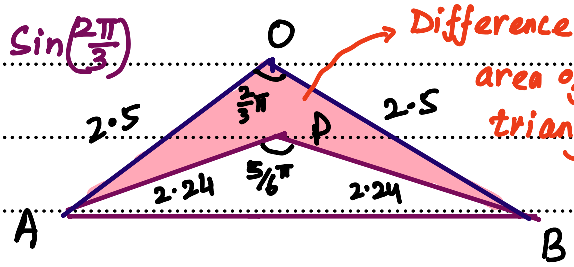
$\frac{28\pi}{15} + \frac{10\pi}{3} = 16.34$

(b) Find the difference in area of the two triangles *AOB* and *APB*, giving your answer correct to 2 decimal places. [2]

Area of Triangle AOB:

$\frac{1}{2} \times 2.5 \times 2.5 \times \sin\left(\frac{2\pi}{3}\right)$

$= 2.71$



Difference in the area of two triangles

Area of Triangle of APB:

$\frac{1}{2} \times 2.24 \times 2.24 \times \sin\left(\frac{5\pi}{6}\right)$

$= 1.2544$

Formula:
Area of triangle
$= \frac{1}{2} ab \sin C$

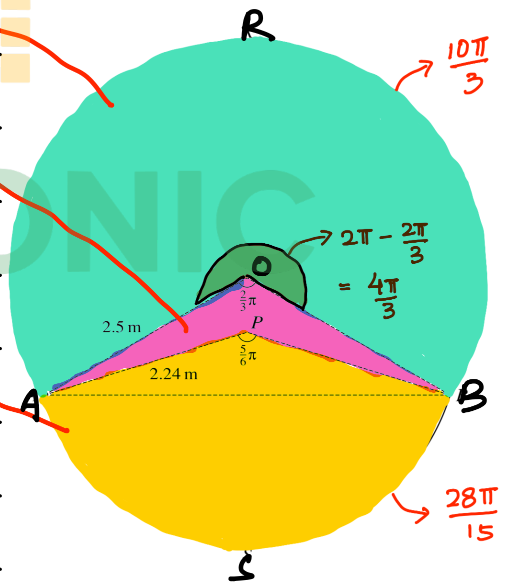
Difference in Area $= 2.71 - 1.2544$

$= 1.45$

(c) Find the area of the cross-section *RASB*, giving your answer correct to 1 decimal place. [3]

Area of Cross Section RASB:

Area of major Sector OARB +
Difference in Area of triangle +
Area of Sector PASB



Area (Major Sector OARB)

$\frac{1}{2} \times 2.5^2 \times \frac{4\pi}{3} = \frac{25\pi}{6}$

Area (Sector PASB)

$\frac{1}{2} \times 2.24^2 \times \frac{5\pi}{6} = 6.57$

Total Area: $\frac{25\pi}{6} + 6.57 + 1.45$

$= 21.1 \text{ m}^2$