Cambridge International AS & A Level

CANDIDATE NAME: SOLVED BY MR. PABITRA

CENTRE NUMBER

CANDIDATE NUMBER

MATHEMATICS **9709/12**

Paper 1 Pure Mathematics 1 **October/November 2024**

1 hour 50 minutes

You must answer on the question paper.

You will need: List of formulae (MF19)

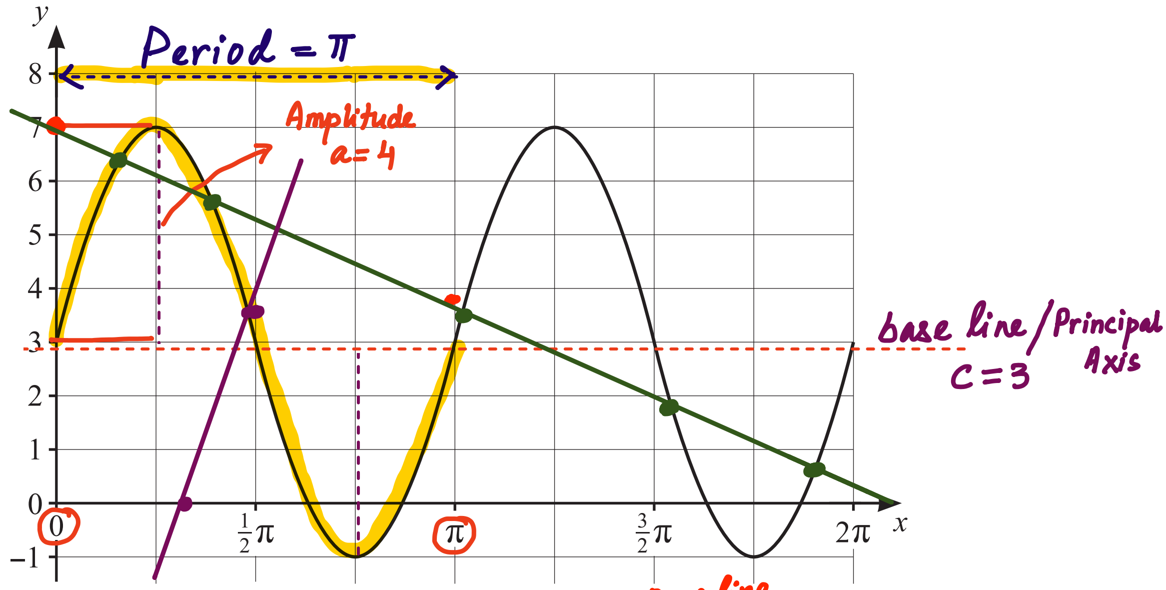
INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

1



Base line

The diagram shows the curve with equation $y = a\sin(bx) + c$ for $0 \leqslant x \leqslant 2\pi$, where a, b and c are positive constants.

Amplitude from base line / Principal Axis

(a) State the values of a, b and c. [3]

Amplitude $(a) = 4$

Base line $(c) = 3$

Period $= \frac{2\pi}{b}$

$\pi = \frac{2\pi}{b}$

$b = \frac{2\pi}{\pi}$

$b = 2$

$y = a\sin(bx) + c$

$y = 4\sin 2x + 3$

(b) For these values of a, b and c, determine the number of solutions in the interval $0 \leqslant x \leqslant 2\pi$ for each of the following equations:

(i) $a\sin(bx) + c = 7 - x$ [1]

$y = 7 - x$

x	0	π
y	7	3.9

$y = 7 - x$ No: of Solutions = 5

Draw the straight line on the curve. The line intersects the curve at 5 Different points.

(ii) $a\sin(bx) + c = 2\pi(x - 1)$. [1]

$y = 2\pi(x-1)$

x	1	$\pi/2$
y	0	3.6

$y = 2\pi(x-1)$ No: of Solutions = 1

Draw the straight line on the curve. The line intersects the curve at 1 point

2 The first term of an arithmetic progression is −20 and the common difference is 5.

(a) Find the sum of the first 20 terms of the progression. [2]

$n = 20 \qquad a = -20 \qquad d = 5$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{20} = \frac{20}{2}[2\times(-20) + (20-1)(5)]$

$= 10[-40 + 95]$

$= 10 \times 55 = 550$

It is given that the sum of the first $2k$ terms is 10 times the sum of the first k terms.

(b) Find the value of k. [3]

$S_{2k} = 10 S_k$

$\cancel{\frac{2k}{2}}[2a + (2k-1)d] = 10 \times \frac{k}{2}[2a + (k-1)d]$

$2\times(-20) + 5(2k-1) = 5[2\times(-20) + 5(k-1)]$

$-40 + 10k - 5 = 5[-40 + 5k - 5]$

$-45 + 10k = 5(-45 + 5k)$

$-45 + 10k = -225 + 25k$

$25k - 10k = -45 + 225$

$15k = 180$

$k = \frac{180}{15}$

$\boxed{k = 12}$

3 The equation of a curve is $y = 2x^2 - 3$. Two points A and B with x-coordinates 2 and $(2 + h)$ respectively lie on the curve.

(a) Find and simplify an expression for the gradient of the chord AB in terms of h. [3]

$y = 2x^2 - 3$

when $x = 2$

$y = 2 \times 2^2 - 3$

$y = 5$

$A\ (2, 5)$

$y = 2x^2 - 3$

when $x = 2 + h$

$y = 2(2+h)^2 - 3$

$y = 2[4 + 4h + h^2] - 3$

$y = 8 + 8h + 2h^2 - 3$

$y = 2h^2 + 8h + 5$

$B\ (2+h,\ 2h^2 + 8h + 5)$

Gradient of AB

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{2h^2 + 8h + 5 - 5}{2 + h - h} = \frac{2h^2 + 8h}{h} = \frac{\cancel{h}(2h + 8)}{\cancel{h}} = \boxed{2h + 8}$$

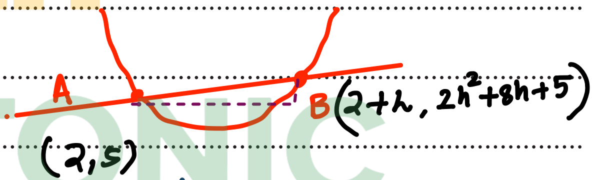
(b) Explain how the gradient of the curve at the point A can be deduced from the answer to part **(a)**, and state the value of this gradient. [2]

As $h \to 0$

(Approaching A)

Chord AB will become tangent @ A.

because the Coordinate of A and B will be Same when $h \to 0$.



Gradient of AB $= 2h + 8$

Gradient at A $= 2(0) + 8 = 8$

4 Find the term independent of x in the expansion of each of the following:

(a) $\left(x+\dfrac{3}{x^2}\right)^6$ [2]

$(a+b)^n = a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \ldots\ldots$

$\left(x+\dfrac{3}{x^2}\right)^6 = x^6 + {}^6C_1 x^5\left(\dfrac{3}{x^2}\right) + \underbrace{{}^6C_2 x^4\left(\dfrac{3}{x^2}\right)^2} + {}^6C_3 x^3\left(\dfrac{3}{x^2}\right)^3 + \ldots\ldots$

$15 \times \cancel{x^4} \times \dfrac{9}{\cancel{x^4}}$

135

(b) $(4x^3-5)\left(x+\dfrac{3}{x^2}\right)^6$. [4]

$\left(x+\dfrac{3}{x^2}\right)^6 = x^6 + {}^6C_1 x^5\left(\dfrac{3}{x^2}\right) + \underbrace{{}^6C_2 x^4\left(\dfrac{3}{x^2}\right)^2}_{135} + \underbrace{{}^6C_3 x^3\left(\dfrac{3}{x^2}\right)^3}_{20 \times x^3 \times \frac{27}{x^6}} + \ldots\ldots$

$540 \times \left(\dfrac{1}{x^3}\right)$

$(4x^3-5)\left(x+\dfrac{3}{x^2}\right)^6 = (4x^3-5)\left(135+\dfrac{540}{x^3}\right)$ ignore all other terms

$= 4x^2 \times \dfrac{540}{x^3} - 5 \times 135$

$= 2160 - 675$

$= 1485$

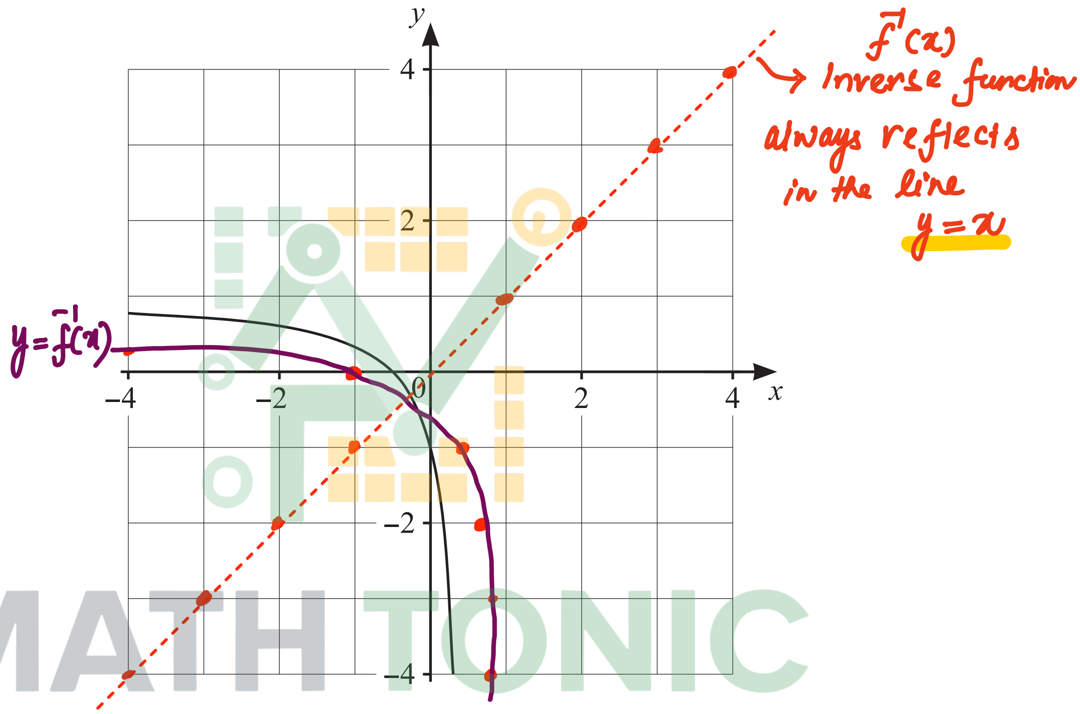
5 The function f is defined by $f(x) = \dfrac{2x+1}{2x-1}$ for $x < \frac{1}{2}$.

(a) **(i)** State the value of $f(-1)$. [1]

$$f(x) = \frac{2x+1}{2x-1}$$

$$f(-1) = \frac{2(-1)+1}{2(-1)-1} = \frac{-1}{-3} = \frac{1}{3}$$

(ii)



The diagram shows the graph of $y = f(x)$. Sketch the graph of $y = f^{-1}(x)$ on this diagram. Show any relevant mirror line. [2]

(iii) Find an expression for $f^{-1}(x)$ and state the domain of the function f^{-1}. [4]

$$y = \frac{2x+1}{2x-1}$$

$$y(2x-1) = 2x+1$$

$$2xy - y = 2x+1$$

$$2xy - 2x = 1+y$$

$$x(2y-2) = 1+y$$

$$x = \frac{1+y}{2y-2}$$

$$f^{-1}(x) = \frac{1+x}{2x-2}$$

$$\boxed{f^{-1}(x) = \frac{1+x}{2(x-1)}}$$

Domain of $f^{-1}(x)$ = Range of $f(x)$

from the graph range of $f(x) < 1$

so, Domain of $f^{-1}(x) < 1$

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From the graph range of $f(x) < 1$

so, Domain of $f^{-1}(x) < 1$

For range always check y axis.

The function g is defined by $g(x) = 3x + 2$ for $x \in \mathbb{R}$.

(b) Solve the equation $f(x) = gf\left(\frac{1}{4}\right)$. [3]

$$f(x) = \frac{2x+1}{2x-1}$$

$$f\left(\tfrac{1}{4}\right) = \frac{2(1/4)+1}{2(1/4)-1} = \frac{3/2}{-1/2} = -3$$

$$g(x) = 3x+2$$

$$gf\left(\tfrac{1}{4}\right) = gf(-3) = 3(-3)+2$$
$$= -9+2$$

$$\boxed{gf(1/4) = -7}$$

$$f(x) = gf(1/4)$$

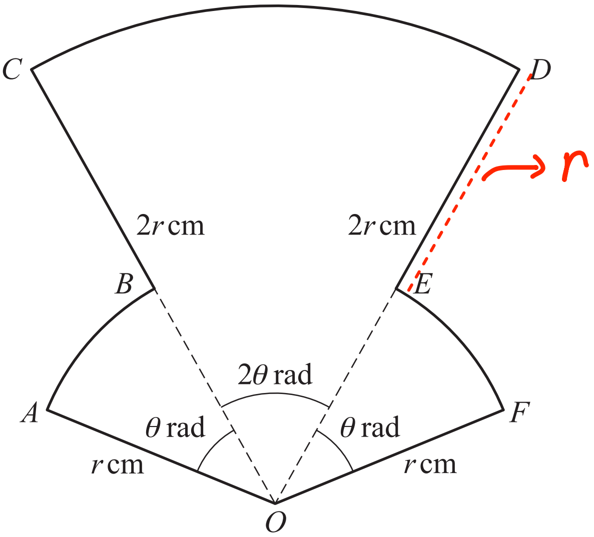
$$\frac{2x+1}{2x-1} = -7$$

$$2x+1 = -7(2x-1)$$

$$2x+1 = -14x+7$$

$$16x = 6 \Rightarrow \boxed{x = \frac{6}{16} = \frac{3}{8}}$$

6



Formula:
Arc length $= r\theta$
Area of Sector $= \frac{1}{2}r^2\theta$

The diagram shows a metal plate $OABCDEF$ consisting of sectors of two circles, each with centre O. The radii of sectors AOB and EOF are r cm and the radius of sector COD is $2r$ cm. Angle AOB = angle $EOF = \theta$ radians and angle $COD = 2\theta$ radians.

It is given that the perimeter of the plate is 14 cm and the area of the plate is 10 cm^2.

Given that $r > \frac{3}{2}$ and $\theta < \frac{3}{4}$, find the values of r and θ. [6]

Perimeter of plate:

OA + Arc AB + BC + Arc CD + DE + Arc EF + OF

$r + r\theta + r + 2r(2\theta) + r + r\theta + r$

$4r + 6r\theta$

Perimeter of plate = 14 cm

$4r + 6r\theta = 14$ —— (i)

Area of the plate:

Area of Sector OAB + Area of Sector OCD + Area of Sector OEF

Area of the plate :

Area of Sector OAB + Area of Sector OCD + Area of Sector OEF

$$\frac{1}{2}r^2\theta \quad + \quad \frac{1}{2}(2r)^2(2\theta) \quad + \quad \frac{1}{2}r^2\theta$$

$$\frac{1}{2}r^2\theta \quad + \quad \frac{1}{2}\times 4r^2 \times 2\theta \quad + \quad \frac{1}{2}r^2\theta$$

$$r^2\theta + 4r^2\theta$$

$$5r^2\theta$$

Area of plate $= 10\ cm^2$

$$5r^2\theta = 10$$

$$r^2\theta = 2$$

$$\theta = \frac{2}{r^2} \quad \text{——— (ii)}$$

Substituting the value of θ into equation (i)

$$4r + 6r\theta = 14$$

$$4r + 6r\left(\frac{2}{r^2}\right) = 14$$

$$4r + \frac{12}{r} = 14$$

$$4r^2 + 12 = 14r$$

$$4r^2 - 14r + 12 = 0$$

$$2r^2 - 7r + 6 = 0$$

$$(2r-3)(r-2) = 0$$

$$2r - 3 = 0$$

$$r = \frac{3}{2}$$

$$r - 2 = 0$$

$$r = 2$$

Given that $r > 3/2$

So, $\boxed{r = 2}$

$$\theta = \frac{2}{r^2} = \frac{2}{2^2} \Rightarrow \boxed{\theta = \frac{1}{2}}$$

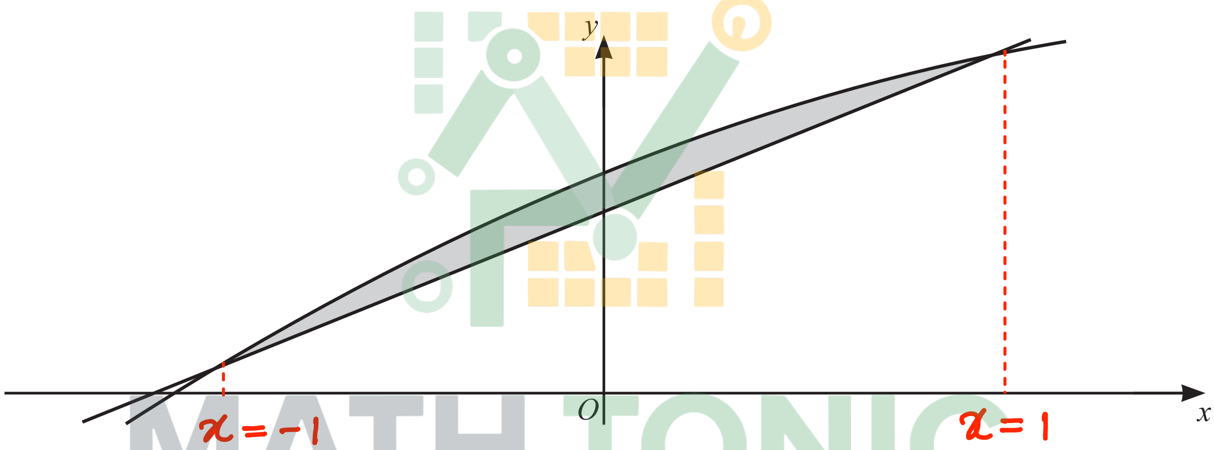
7 (a) By expressing $-2x^2+8x+11$ in the form $-a(x-b)^2+c$, where a, b and c are positive integers, find the coordinates of the vertex of the graph with equation $y=-2x^2+8x+11$. [3]

$-2x^2+8x+11$
$= -2(x^2-4x)+11$
$= -2[(x-2)^2-2^2]+11$
$= -2[(x-2)^2-4]+11$
$= -2(x-2)^2+8+11$
$= -2(x-2)^2+19$

$-2(x-2)^2+19$

Vertex :
$(2, 19)$

(b)



The diagram shows part of the curve with equation $y=-2x^2+8x+11$ and the line with equation $y=8x+9$.

Find the area of the shaded region. [5]

For intersection of line and Curves Solve as Simultaneous equations :

$y = -2x^2+8x+11 \qquad y=8x+9$

$-2x^2+8x+11 = 8x+9$

$-2x^2+8x-8x = 9-11$

$-2x^2 = -2$

$x^2 = 1$

$x = \pm\sqrt{1} \Rightarrow x = \pm 1$

Shaded Area $= \int \text{upper} - \text{lower}$

$$= \int_{-1}^{1} \left[(-2x^2 + 8x + 11) - (8x + 9) \right] dx$$

$$= \int_{-1}^{1} (-2x^2 + 8x + 11 - 8x - 9)\, dx$$

$$= \int_{-1}^{1} (-2x^2 + 2)\, dx$$

$$= \left[\frac{-2x^3}{3} + 2x \right]_{-1}^{1}$$

$$= \left[\frac{-2(1)^3}{3} + 2(1) \right] - \left[\frac{-2(-1)^3}{3} + 2(-1) \right]$$

$$= \left(\frac{-2}{3} + 2 \right) - \left(\frac{2}{3} - 2 \right)$$

$$= \frac{4}{3} + \frac{4}{3}$$

$$= \frac{8}{3}$$

8 The equation of a circle is $x^2+y^2+px+2y+q=0$, where p and q are constants.

(a) Express the equation in the form $(x-a)^2+(y-b)^2=r^2$, where a is to be given in terms of p and r^2 is to be given in terms of p and q. [2]

$x^2+y^2+px+2y+q=0$

$x^2+px+y^2+2y=-q$

$\left(x+\frac{p}{2}\right)^2-\left(\frac{p}{2}\right)^2+(y+1)^2-1^2=-q$ using completing the square method.

$\left(x+\frac{p}{2}\right)^2-\frac{p^2}{4}+(y+1)^2-1=-q$

$\left(x+\frac{p}{2}\right)^2+(y+1)^2=\frac{p^2}{4}+1-q$

$a=\frac{-p}{2}$, $b=-1$, $r^2=\frac{p^2}{4}+1-q$

The line with equation $x+2y=10$ is the tangent to the circle at the point $A(4, 3)$.

(b) (i) Find the equation of the normal to the circle at the point A. [3]

Tangent : $x+2y=10$

$2y=-x+10$

$y=-\frac{1}{2}x+5$ $\qquad$ $y=mx+c$ gradient

Gradient of tangent $=-\frac{1}{2}$

For perpendicular lines, $m_1\times m_2=-1$

Gradient of Normal $=2$

Equation of Normal :

$y-y_1=m(x-x_1)$

at $A(4,3)$ $\quad$ $y-3=2(x-4)$

$y-3=2x-8$

$y=2x-8+3$

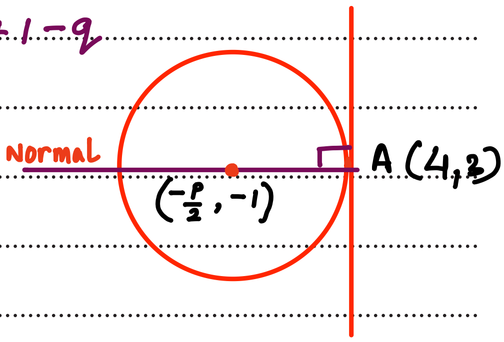
$\boxed{y=2x-5}$

(ii) Find the values of p and q. [5]

Equation of Circle :

$$\left(x+\frac{p}{2}\right)^2+(y+1)^2=\frac{p^2}{4}+1-q$$

Normal passes through the Centre of Circle.



At $C\left(-\frac{p}{2},-1\right)$

$$y=2x-5$$
$$-1=2\left(\frac{-p}{2}\right)-5$$
$$-1=-p-5$$
$$p=-5+1$$
$$\boxed{p=-4}$$

Centre $\left(-\frac{p}{2},-1\right)$

$-\frac{(-4)}{2}, -1$

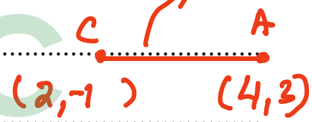
$C\,(2,-1)$

Equation of Circle :

$$\left(x+\frac{p}{2}\right)^2+(y+1)^2=\underbrace{\frac{p^2}{4}+1-q}_{r^2}$$

$$r^2=\frac{p^2}{4}+1-q$$
$$r^2=\frac{(-4)^2}{4}+1-q$$
$$r^2=4+1-q$$
$$r^2=5-q$$

radius.



radius : $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$\sqrt{(4-2)^2+(3+1)^2}$

$\sqrt{2^2+4^2}$

$r=\sqrt{20}$

$r^2=20$

$$5-q=20$$
$$q=5-20$$
$$\boxed{q=-15}$$

9 The equation of a curve is $y = \frac{1}{2}k^2x^2 - 2kx + 2$ and the equation of a line is $y = kx + p$, where k and p are constants with $0 < k < 1$.

(a) It is given that one of the points of intersection of the curve and the line has coordinates $\left(\frac{5}{2}, \frac{1}{2}\right)$.

Find the values of k and p, and find the coordinates of the other point of intersection. [7]

Given that : One of the point of intersection $\left(\frac{5}{2}, \frac{1}{2}\right)$

Equation of the Curve :

At $\left(\frac{5}{2}, \frac{1}{2}\right)$

$$y = \frac{1}{2}k^2x^2 - 2kx + 2$$

$$\frac{1}{2} = \frac{1}{2}k^2\left(\frac{5}{2}\right)^2 - 2k\left(\frac{5}{2}\right) + 2$$

$$\frac{1}{2} = \frac{25k^2}{8} - 5k + 2$$

Multiplying 8 with each term

$$4 = 25k^2 - 40k + 16$$

$$25k^2 - 40k + 12 = 0$$

$$(5k-2)(5k-6) = 0$$

$5k - 2 = 0$ $\quad$ $5k - 6 = 0$

$k = \frac{2}{5}$ $\quad$ $k = \frac{6}{5}$

Given that : $0 < k < 1$

$$\boxed{k = \frac{2}{5}}$$

Equation of line :

At $\left(\frac{5}{2}, \frac{1}{2}\right)$

$$y = kx + p$$

$$\frac{1}{2} = \left(\frac{2}{5} \times \frac{5}{2}\right) + p$$

$$\frac{1}{2} = 1 + p$$

$$\boxed{p = -\frac{1}{2}}$$

$k = \frac{2}{5}$, $p = -\frac{1}{2}$

Equation of the Curve :

$$y = \frac{1}{2}k^2x^2 - 2kx + 2$$

$$y = \frac{1}{2}\left(\frac{2}{5}\right)^2x^2 - 2\left(\frac{2}{5}\right)x + 2$$

$$y = \frac{2}{25}x^2 - \frac{4}{5}x + 2$$

Equation of line :

$$y = kx + p$$

$$y = \frac{2}{5}x - \frac{1}{2}$$

$$\frac{2}{25}x^2 - \frac{4}{5}x + 2 = \frac{2}{5}x - \frac{1}{2}$$

$$\frac{2}{25}x^2 - \frac{4}{5}x - \frac{2}{5}x + 2 + \frac{1}{2} = 0$$

multiply 25 in each term

$$2x^2 - 20x - 10x + 50 + \frac{25}{2} = 0$$

$$2x^2 - 30x + \frac{125}{2} = 0$$

multiply 2 in each term

$$4x^2 - 60x + 125 = 0$$

$$(2x-5)(2x-25) = 0$$

$$x = 5/2, \quad x = 25/2$$

For $x = \frac{25}{2}$

$$y = \frac{2}{5}x - \frac{1}{2}$$

$$y = \frac{2}{5}\left(\frac{25}{2}\right) - \frac{1}{2}$$

$$y = 5 - \frac{1}{2} = \frac{9}{2}$$

Other point of intersection:

$$\boxed{\frac{25}{2}, \frac{9}{2}}$$

(b) It is given instead that the line and the curve do **not** intersect.

Find the set of possible values of p. [3]

$$\frac{1}{2}k^2x^2 - 2kx + 2 = kx + p$$

$$\frac{1}{2}k^2x^2 - 2kx - kx + 2 - p = 0$$

$$\underbrace{\frac{1}{2}k^2}_{a}x^2 \underbrace{- 3k}_{b}x + \underbrace{(2-p)}_{c} = 0$$

For no intersection:

$$b^2 - 4ac < 0$$

$$(-3k)^2 - 4\left(\frac{1}{2}k^2\right)(2-p) < 0$$

$$9k^2 - (2k^2)(2-p) < 0$$

$$9k^2 - 4k^2 + 2k^2p < 0$$

$$5k^2 + 2k^2p < 0$$

$$k^2(5+2p) < 0$$

+ −

This is always positive because of square term

Output should be negative

$$5 + 2p < 0$$

$$2p < -5$$

$$\boxed{p < -\frac{5}{2}}$$

10 A function f with domain $x > 0$ is such that $f'(x) = 8(2x-3)^{\frac{1}{3}} - 10x^{\frac{2}{3}}$. It is given that the curve with equation $y = f(x)$ passes through the point (1, 0).

(a) Find the equation of the normal to the curve at the point (1, 0). [3]

$f'(x) = 8(2x-3)^{1/3} - 10x^{2/3}$

At (1, 0) $f'(1) = 8(2\times1-3)^{1/3} - 10\times1^{2/3}$

$f'(1) = -18$

gradient of Curve $= -18$

gradient of normal $= \frac{1}{18}$

$(m_1 \times m_2 = -1)$

Equation of Normal:

$y - y_1 = m(x - x_1)$

$y - 0 = \frac{1}{18}(x-1)$

$\boxed{y = \frac{1}{18}x - \frac{1}{18}}$

(b) Find f(x). [4]

$f'(x) = 8(2x-3)^{1/3} - 10x^{2/3}$

$f(x) = \int \left(8(2x-3)^{1/3} - 10x^{2/3}\right) dx$

$f(x) = \frac{8(2x-3)^{4/3}}{4/3 \times 2} - \frac{10x^{5/3}}{5/3} + C$

$f(x) = 3(2x-3)^{4/3} - 6x^{5/3} + C$

$y = 3(2x-3)^{4/3} - 6x^{5/3} + C$

At (1, 0)

$0 = 3(2\times1-3)^{4/3} - 6\times1^{5/3} + C$

$0 = 3 - 6 + C$

$C = 3$

$\boxed{f(x) = 3(2x-3)^{4/3} - 6x^{5/3} + 3}$

It is given that the equation $f'(x) = 0$ can be expressed in the form

$$125x^2 - 128x + 192 = 0.$$

(c) Determine, making your reasoning clear, whether f is an increasing function, a decreasing function or neither. [3]

$125x^2 - 128x + 192 = 0$

Rule:
If
$\frac{dy}{dx} < 0$
decreasing function.

$125x^2 - 128x + 192$

$125\left[x^2 - \frac{128}{125}x\right] + 192$

$125\left[\left(x - \frac{64}{125}\right)^2 - \left(\frac{64}{125}\right)^2\right] + 192$

$125\left(x - \frac{64}{125}\right)^2 - 125\times\left(\frac{64}{125}\right)^2 + 192$

$125\underbrace{\left(x - \frac{64}{125}\right)^2}_{\text{square term} \geqslant 0} \underbrace{- \frac{19904}{125}}_{< 0}$

For all value of x,

$\left(x - \frac{64}{125}\right)^2 \geqslant 0$ $\frac{dy}{dx} < 0$

It is an decreasing function.

Other method: $\underbrace{125}_{a}x^2 \underbrace{- 128}_{b}x \underbrace{+ 192}_{c} = 0$

$b^2 - 4ac$

$(-128)^2 - 4(125)(192)$

$-79616 < 0$

Which means there are no turning points. So, no solutions can be found for $125x^2 - 128x + 192 = 0$

$f'(x) = 125x^2 - 128x + 192$

$f'(10) = -25.85$

$f'(100) = -168.89$

when $\frac{dy}{dx} < 0$

It is a decreasing function

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

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